THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 27, 1904, extending through the usual morning and afternoon sessions. The first part of the afternoon was devoted to a joint session with the American Physical Society for the purpose of hearing the presidential address of Professor A. G. Webster on "Some practical aspects of the relations between physics and mathematics." This address will be published in the *Physical Review*.

The following forty-one members attended the several sessions:

Dr. Grace Andrews, Professor Joseph Bowden, Professor F. N. Cole, Miss L. D. Cummings, Mr. R. F. Deimel, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findley, Professor T. S. Fiske, Mr. C. S. Forbes, Dr. A. S. Gale, Miss Ida Griffiths, Professor James Harkness, Professor H. E. Hawkes, Mr. E. A. Hook, Dr. J. G. Hun, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Dr. Emilie N. Martin, Dr. C. M. Mason, Professor M. I. Pupin, Miss Virginia Ragsdale, Miss I. M. Schottenfels, Professor Henry Taber, Professor J. H. Tanner, Professor H. D. Thompson, Miss M. E. Trueblood, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor J. B. Webb, Professor A. G. Webster, Miss E. C. Willaims, Dr. E. B. Wilson, Dr. Ruth G. Wood, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair during the regular sessions and the joint session with the Physical Society. The Council announced the election of the following persons to membership in the Society: Mr. E. P. R. Duval, Harvard University; Professor G. A. Goodenough, University of Illinois; Mr. H. C. Harvey, State Normal School, Kirksville, Mo.; Dr. J. G. Hun, Princeton University; Dr. T. P. Running, University of Michigan. Nine applications for membership were received.

Professor E. H. Moore was re-elected a member of the Editorial Committee of the *Transactions* for a term of three years.

The following papers were presented at this meeting:

(1) Dr. WILLIAM FINDLAY: "The Sylow subgroups of the symmetric group."

(2) Dr. L. P. EISENHART: "Three particular systems of lines on a surface."

(3) Professor JOSEPH BOWDEN: "The definitions of sine and cosine."

(4) Professor H. E. HAWKES: "On quaternion number systems."

(5) Professor L. E. DICKSON: "On the subgroups of order a power of p in the linear homogeneous and fractional groups in the $GF[p^n]$."

(6) Dr. C. M. MASON: "On the solutions of $\Delta u + \lambda A(x, y)u = f(x, y)$ which satisfy prescribed boundary conditions."

(7) Professor F. N. COLE: "The groups of order p^3q^3."

(8) Dr. EDWARD KASNER: "Galileo on the concept of infinity."

(9) Professor E. W. BROWN: "On the smaller perturbations of the lunar arguments."

(10) Professor E. B. VAN VLECK: "On the convergence of algebraic continued fractions in which the coefficients have a limiting form."

(11) Professor HENRY TABER: "Hypercomplex number systems."

(12) Dr. EDWARD KASNER: "On the geometry of ordinary differential equations."

(13) Miss I. M. SCHOTTENFELS: "On a theory of functions related to a hypercomplex number system in two units."

(14) Mr. G. D. BIRKHOFF: "A general remainder theorem."

Mr. Birkhoff was introduced by Professor Osgood. In the absence of the authors the papers of Professor Dickson, Professor Brown and Professor Taber were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Sylow proved that a group of order g contains one and only one set of conjugate subgroups of order the highest power of p (any prime) contained in g, which may thus be called its Sylow subgroups. Dr. Findlay proves that in case n is not a

power of p the Sylow subgroup of the symmetric group of degree n is the direct product of Sylow subgroups of symmetric groups on distinct sets of letters, the number in each set being a power of p, and thereby reduces the discussion to the case $n = p^\alpha$. He then obtains the generators of a Sylow subgroup of degree p^α, the number of them being the same as the exponent of p in the order of the group. The complex classification of the p^α letters used in the definition of the generators is proved to furnish a complete analysis of the imprimitivity of the group. The various groups of substitutions upon the systems of imprimitivity induced by the substitutions of the group are seen to be Sylow subgroup of symmetric groups whose degrees are lower powers of p; they are also quotient groups under the initial group of a series of invariant subgroups possessing important properties. The commutator series of subgroups is obtained and also all subgroups which may be considered as Sylow subgroups of symmetric groups. Enumerations are made of the substitutions of orders p and p^α, and the conjugacy of the latter is discussed. The subgroup consisting of the invariant substitutions is of order p. The author finds the number of Sylow subgroups of the symmetric group and classifies them according to their systems of imprimitivity.

2. Cifarelli has shown that if one of two quadratic differential forms is definite there exists a real transformation of variables such that the definite form can be reduced to $a_1du^2 + a_3dv^2$ and the second form to $b_1du^2 + 2b_2dudv + b_3dv^2$ with the relation

$$a_1/b_1 = a_3/b_3.$$

Dr. Eisenhart has applied this transformation to pairs of the three fundamental quadratic forms associated with every surface and determining it. It is found that there are three cases, giving rise to as many systems of lines $v = \text{const.}$, $u = \text{const.}$ upon the surface. For all these systems the radii of normal curvature and geodesic torsion in the directions of the curves can be expressed in simple forms in terms of the principal radii of the surface.

The lines in one system are orthogonal to one another and form equal angles with the lines of curvature; a second system is the unique conjugate system whose included angle is bisected by the lines of curvature; and the third system is com-

posed of the conjugates of the first and they are represented on the sphere by orthogonal lines. The directions of the curves of the second system form angles with the directions of lines of curvature in one family which are less than the angles of the first system and greater than those of the conjugates of the latter. For the curves of the first system to be isothermal, or those of the second to be isothermal-conjugate, or those of the third to have isothermal spherical representation, the lines of curvature must have the same property in each case. Only certain systems upon the sphere can represent the second system, in which case the surface is unique and determined by quadratures; whereas any orthogonal system upon the sphere will represent the third system and there is a double infinity of surfaces corresponding, whose determination requires the integration of a partial differential equation of the second order and quadratures.

3. Professor Bowden proved from the definitions $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ that if, when θ is real, the rectangular coördinates of any point in a plane are x, y and its corresponding polar coördinates r, θ, where θ is the number of radians in the vectorial angle, then $\sin\theta = y/r$ and $\cos\theta = x/r$.

4. In a paper in the *Mathematische Annalen* (volume 58, page 361), Professor Hawkes solved the general enumeration problem for non-quaternion number systems. The present paper treats the same problem for quaternion systems. A normal form is first obtained to which any quaternion system may be reduced and by an inspection of which the characteristic properties of the system may be determined. A simple method of writing down all distinct systems of a given order in this normal form is then given, thus completing the solution of the general enumeration problem of hypercomplex number systems which are associative and have a modulus.

5. The paper by Professor Dickson considers the subgroups of order a power of p of the general linear homogeneous, special linear homogeneous, and linear fractional groups on m variables with coefficients in the $GF[p^n]$. Some of the results for the third group may be noted. A subgroup of order the highest possible power of p, namely, $p^{\mu n}$, $\mu \equiv \frac{1}{2}m(m-1)$, is transformed into

itself by exactly $(p^n-1)^{m-1}p^{\mu n}/d$ operators of $LF(m, p^n)$, where d is the greatest common divisor of m and p^n-1. The orders of the largest subgroups of $LF(4, p)$ transforming subgroups of order p^5 into themselves are

$$(p-1)^2p^6/d, \qquad a(p-1)^2p^6/d,$$
$$a(p-1)p^6/d, \qquad (p^2-1)(p-1)^2p^6/d,$$

where $a=2$ if $p>2$, $a=1$ if $p=2$. For $LF(3, p)$ and subgroups of order p^2, the numbers are $(p^2-1)(p-1)p^3/d$ and $(p-1)p^3$.

6. Dr. Mason applied to the three boundary value problems of the differential equation $\Delta u+\lambda A(x, y)u=f(x, y)$ a method derived by Tredholm for solving certain functional equations. General theorems were deduced, and with their aid the existence and minimal characteristics of an infinite series of harmonic functions of the equation was proved by considering the equation as the necessary condition of a calculus of variations problem. The paper will appear in the *Journal de Mathématiques*.

7. Professor Cole discussed the groups of order $p^\beta q^\beta$, where p and q are different prime numbers, and showed that these groups are all compound and therefore solvable. Certain useful auxiliary theorems of general application to groups of order $p^\alpha q^\beta$ were also established. This paper appeared in the April number of the *Transactions*.

8. Dr. Kasner called attention to a significant passage from Galileo's Discorsi e dimonstrazione matematiche (Leyden, 1638) which appears to have been neglected in tracing the development of the modern concept of infinity.

9. The motions of the lunar perigee and node are affected by other forces than the sun's attraction. Amongst these, the principal are the effects caused by direct and indirect planetary action and by the non-spherical form of the earth. A general method is given in Professor Brown's paper by which all such actions can be very easily and accurately accounted for, the work of actually computing them being reduced to a few substitutions of numerical values in simple formulæ. The necessary data are given and the results for the principal perturbations obtained. This paper is preliminary to a general

investigation of the complete perturbations caused in the motion of the moon by attractions other than those of the earth and sun, considered as particles.

10. Padé in his thesis has shown that to every Maclaurin power series there correspond, in what may be called the normal case, three classes of regular continued fractions. These, after certain opening irregularities which are permissible in the first one or two partial fractions, have the forms respectively

$$\text{(I)} \qquad + \cfrac{a_2 x}{1} + \cfrac{a_3 x}{1} + \cfrac{a_4 x}{1} + \cdots,$$

$$\text{(II)} \qquad + \cfrac{a_2 x^2}{1 + b_2 x} + \cfrac{a_3 x^2}{1 + b_3 x} + \cfrac{a_4 x^2}{1 + b_4 x} + \cdots,$$

$$\text{(III)} \qquad + \cfrac{a_2 x}{1 + b_2 x} + \cfrac{a_3 x}{1 + b_3 x} + \cfrac{a_4 x}{1 + b_4 x} + \cdots.$$

The paper of Professor Van Vleck contained an investigation of the convergence of these classes of continued fractions upon the assumption of the existence of limiting values for the coefficients a_n and b_n.

The result obtained for the first class of continued fractions is as follows: If $\lim_{n=\infty} a_n = k$, the continued fraction (I) converges over the entire plane of x with the exception (1) of the whole or a part of a cut drawn from $x = -1/4k$ to $x = \infty$ with an argument equal to that of the vector from the origin to $x = -1/4k$, and, possibly also, (2) of a set of isolated points p_1, p_2, p_3, $\cdots$ The limit of the continued fraction within the plane thus cut is holomorphic at every point except p_1, p_2, p_3, $\cdots$, which are poles.

This theorem had been previously demonstrated by the author with certain restrictions in the *Annals of Mathematics*, but the employment of a new method has given a general demonstration. For class (II) a like result holds, the cut being the segment of a straight line or the arc of a circle. These conclusions are not affected by the introduction of a finite number of irregularities (*i. e.*, partial numerators and denominators of higher degrees) into the two continued fractions. The third class of continued fractions, however, exhibit certain anomalies not shown by the first two.

11. The object of Professor Taber's paper is to establish Benjamin Peirce's method for the determination of hypercomplex number systems in a given number of units without recourse to the theory of transformation groups, and by purely algebraic methods. This is accomplished by the extension to number systems in general of the scalar function of quaternions, defined as follows: If c_{ijk} are the constants of multiplication of the system,

$$S\sum_{i=1}^{n} a_i e_i = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ijj}.$$

Peirce's method is extended to real hypercomplex number systems; and it is shown that, in the first group $(e_1, e_2, \cdots e_n)$ with respect to the idempotent number e_n, the remaining units can be so selected that p of them shall be nilpotent, forming a nilpotent system by themselves, the remaining units e_{p+1}, e_{p+2}, $\cdots$, e_{n-1} satisfying the condition

$$e_{p+1}^2 = e_{p+2}^2 = \cdots = e_{n-1}^2 = -e_n^2.$$

And we have either $p = n-1$, $p = n-2$, or $p = n-4$; and thus the third class of units are in number either 0, 1, or 3. When there are three units we obtain a system of which real quaternions is a special case.

12. In connection with a given differential equation $y' = f(x, y)$, it is of interest to consider the *derived* equation $y' = -f_x/f_y$. Geometrically, the system of curves defined by the latter equation may be termed the *slope* curves of the system defined by the original equation. The first question considered by Dr. Kasner is the relation between the infinite number of distinct systems which have the same slope system; the result is of interest in connection with graphical integration. The next topic is an examination of the familiar classes of differential equations with reference to their slope systems; for example, it is shown that the derived equation of a linear equation is itself linear. The rest of the paper is devoted to systems of curves for which the relation between original and derived systems is mutual, so that the original system may be regarded as the derived of the second system.

13. Miss Schottenfels's paper establishes a function theory which has immediate application to the hypercomplex number system in two units e_1, e_2, where $e_1^2 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_2^2 = 0$.

The necessary and sufficient relations between the functional forms to ensure the required functionality are derived; the terms limit, differentiation, and integration are defined: and the exponential and certain trigonometric functions are discussed.

14. Assuming that n values of a function $f(x)$ and its derivatives are known

$$(1) \qquad f^{(\kappa_\iota)} x_\iota = a_\iota \qquad (\iota = 1, 2, \cdots, n;\ \kappa_\iota \leqq n-1)$$

with certain continuity conditions on $f(x), f'(x), \cdots, f^{(n)}(x)$ and a simple restriction on the system of pairs (κ_ι, x_ι), Mr. Birkhoff derives a general remainder theorem

$$(2) \quad f^{(\kappa_0)}(x_0) = F^{(\kappa_0)}(x_0) + \frac{f^{(n)}(\xi_1) - \dfrac{E}{1-E}[f^{(n)}(\xi_\iota) - f^{(n)}(\xi_1)]}{n!} \cdot \psi^{(\kappa_0)}(x_0)$$

for an arbitrary derivative (κ_0) at an arbitrary point x_0, E not independent of (κ_0, x_0). All the known remainder theorems come directly under the case $\kappa_0 = 0$, $E = 0$. The application to a general mean value theorem and the problems of interpolation follow.

F. N. COLE.

COLUMBIA UNIVERSITY.

REPORT ON THE REQUIREMENTS FOR THE MASTER'S DEGREE.

PRESENTED TO THE CHICAGO SECTION OF THE AMERICAN MATHEMATICAL SOCIETY AND RECOMMENDED FOR PUBLICATION, JANUARY 1, 1904.

AT the Christmas meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY, held January 2 and 3, 1902, a committee was appointed to consider and report a scheme of requirements for candidates proceeding to their second degree, with mathematics as their major subject. It was thought that a mathematical programme generally adopted in the central west would facilitate the migration of students from one institution to another, thus permitting students to take portions of

their mathematical work at different institutions; that it would tend to preserve the value of the master's degree by making it represent equivalent attainments; and that it would enable the smaller colleges better to arrange their elective studies, so that students intending to make mathematics a specialty could, by a proper selection, obtain sufficient mathematical training to enter at once after graduation upon work which might be counted toward the master's degree.

This committee through its chairman sent a circular letter of inquiry to seventeen institutions of the central West, including all of the state universities from Ohio to Colorado, and others which may be regarded as of the same rank, or in which the mathematical department is especially strong. This letter invited a free discussion both as to the actual practice in these institutions and as to what might seem desirable as a mathematical programme covering both the undergraduate work and that leading to the second degree. As a basis of undergraduate work it was assumed that the student enters college with a thorough knowledge of algebra through quadratic equations and with plane and solid geometry. It was also assumed that the master's degree is to be conferred upon the successful completion of one year of *graduate* work, and that the college course covers four years of academic training. The scope of the inquiry is shown by the questions asked. They were as follows:

1. What should be the *minimum* acceptable time devoted to undergraduate mathematics?

2. What subjects should be included in the undergraduate course in pure mathematics? How much time should be devoted to each?

3. Should applied mathematics be required? What subjects, and how much?

4. Should French and German, one or both, be required? If only one, which one?

5. Should the candidate for the master's degree be required to pass a more rigorous examination than is required of the undergraduate?

6. Should a thesis be required?

7. Should the graduate work leading to the master's degree be confined to one, two, or three departments?

In order to make it more easily interpreted in the particular system of accrediting subjects at the several institutions, the

committee has formulated its report in terms of recitation hours; thus a subject coming five times a week for one year of 36 weeks is assigned 180 hours.

After canvassing the replies from the institutions mentioned above, the committee submits the following mathematical programme as giving the *minimum* preparations which should be accepted as sufficient for entering upon graduate work. It further reports a programme of graduate work, which is recommended as the minimum basis for granting the second degree.

1. The minimum time devoted to mathematics in the undergraduate course should be 540 hours. This means that at least one-fourth of the student's undergraduate work should be devoted to the study of mathematics, if he expects to secure his master's degree in mathematics after one year of resident graduate study. Whether this amount, or more, be taken, it is recommended that it be distributed throughout the four years of the undergraduate course, to the end that the continuity of his mathematical study may not be disturbed as the student passes from his undergraduate to his graduate work.

2. The 540 hours mentioned above should be distributed substantially as follows:

I. College algebra, including the elements of determinants; trigonometry; and analytic geometry, including an introduction to solid analytic geometry. 180 hours.

II. Differential and integral calculus, differential equations, and mechanics. 225 hours.

III. Advanced algebra, including the general theory of equations; modern analytic and synthetic geometry, or solid analytic geometry. 135 hours.

3. The undergraduate instruction, especially in its earlier stages, should, by well chosen illustrations and applications, be kept in close contact with the physical sciences. Logical statement and proofs of theorems should always be delayed until their meaning is made familiar by application to special cases, and the problems considered for such purposes should be such as arise naturally out of concrete experiences. Whenever possible, a physical or a graphical solution of the problem should be given. In all mathematical teaching a clear comprehension of ideas and their correlation are of prime importance. In the earlier stages of any subject these can in general be best

secured through the physical senses and by a method analogous to that of the general science laboratories. Abstract analysis should arise naturally in the effort to devise the most precise, clear, and convenient method of formulating and expressing the results of experiments and observations; and only in the later stages should it become an object of thought and investigation.

4. Upon the basis of the preparation indicated above, it is the opinion of the committee that the work leading to the master's degree should include at least 270 hours selected from the following groups of subjects. The selection should in any case include subjects from at least two of the three groups.

I. GEOMETRY.

Projective geometry.
Modern analytic geometry, algebraic curves and surfaces.
Application of calculus to twisted curves and surfaces. (Differential geometry.)
Solid analytic geometry.
Descriptive geometry (Darstellende Geometrie).

II. ANALYSIS.

Theory of equations.
Advanced calculus.
Theory of functions.
Differential equations.
Theory of numbers.
Invariants.

III. APPLIED MATHEMATICS.

Analytic mechanics.
Mathematical astronomy.
Mathematical physics.
Mathematical theory of probability.

This arrangement provides that the candidate, in case he enters upon his graduate work with the minimum mathematical preparation indicated above, and hence having a broader *general* training in his undergraduate course, should be required to take at least two thirds of his year's graduate work in pure and applied mathematics. On the other hand, as it may be assumed that the student entering upon his graduate work with a

larger mathematical credit than this has had less *general* training in his undergraduate course, it is the opinion of the committee that he should be permitted to take as much as one half of his work in departments other than mathematical, providing, however, that the minimum requirement in mathematics indicated above is also fulfilled.

5. The candidate for a second degree should enter upon his graduate work with a sufficient knowledge of German to enable him to read easily mathematical works in that language. The committee thinks it very desirable that he should have also a reading knowledge of French.

6. It is the opinion of a large majority of the institutions with which the committee has corresponded, as well as the opinion of the committee, that in all cases a thesis should be required. This may not, and in most cases cannot, be in the strictest sense original research. It is, however, a convenient way of testing the student's ability to carry on independent investigation, and gives him an excellent preparation for research work later.

7. The examination of the candidate for a master's degree should be distinguished as to rigor and perhaps as to character from the ordinary term by term examination of the undergraduate student. Two plans are proposed, either of which, or a combination of the two, is commended.

(*a*) That the candidate be permitted to take the term by term examinations, as in the case of undergraduates, but that he should be expected to earn a higher grade than would entitle him to pass in case he were an undergraduate.

(*b*) That he be required to pass a final examination at the time of taking his degree covering all of the subjects included in his graduate course.

The committee finds a great diversity of practice as to the amount of mathematics which is made the basis of graduate work at the various institutions mentioned at the beginning of this report. It believes, however, that the programme outlined above is within the reach of all. Furthermore, upon comparison with the requirements at several of the leading American universities, it was found that the minimum requirements which are here recommended are in substantial accord with the minimum requirements for the master's degree at these institutions.

It is the thought of the committee that, as meeting the above requirements in preparation for graduate work, the work of no

institution claiming college grade should be accepted unless the major part of the work in preparation has been taken under the direction of mathematical instructors who have themselves done the equivalent of the work here outlined for the master's degree in mathematics.

By the adoption of the above mathematical programme by the institutions represented in the Chicago Section, it is believed that a substantial service would be rendered the study of mathematics in the central west, not only by securing the advantage of uniformity in granting the second degree, but by elevating in some cases the standard upon which the degree is granted, and perhaps more than all else by giving to the small college, aiming to prepare for graduate work, a standard by which it may best arrange its elective system.

C. A. WALDO, *Chairman*,
E. J. TOWNSEND,
OSKAR BOLZA,
Committee.

ON THE SUBGROUPS OF ORDER A POWER OF p IN THE LINEAR HOMOGENEOUS AND FRACTIONAL GROUPS IN THE $GF[p^n]$.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, February 27, 1904.)

1. THIS paper relates primarily to the subgroups G of order the highest power of p in the m-ary general and special linear homogeneous groups and the linear fractional group in the $GF[p^n]$. For the latter groups the question of the minimum index of subgroups is of importance in various applications of group theory. A knowledge of the properties of G contributes materially towards an answer to this question, as will be shown in a subsequent note.

2. *Notations.* — The general linear homogeneous group $GLH(m, p^n)$ of all m-ary transformations in the $GF[p^n]$ has the order

$$\Omega_{m,p^n} = (p^{nm} - 1)(p^{n(m-1)} - 1) \cdots (p^n - 1)p^{n\mu},$$

$$[\mu \equiv \tfrac{1}{2}m(m-1)].$$

Its transformations of determinant unity form the special linear homogeneous group $SLH(m, p^n)$ of order $\Omega_{m,p^n} \div (p^n - 1)$. Forming the quotient group of the latter by the group of the transformations which multiply each variable by the same mark, we obtain the linear fractional group $LF(m, p^n)$ of order

$$\omega_{m,p^n} = \frac{1}{d}(p^{nm} - 1)(p^{n(m-1)} - 1)\cdots(p^{2n} - 1)p^{\mu n},$$

where d is the greatest common divisor of m and $p^n - 1$. Except for $(m, p^n) = (2, 2)$ and $(2, 3)$, $LF(m, p^n)$ is simple.

In this paper, (α_{ij}) denotes a general matrix of m^2 elements α_{ij}, while $[\alpha_{ij}]$ denotes a matrix of the special form

$$(1) \qquad [\alpha_{ij}] \equiv \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \alpha_{m4} & \cdots & \alpha_{mm-1} & 1 \end{vmatrix},$$

with every $\alpha_{ii} = 1$, $\alpha_{ij} = 0$, $(j > i)$.

The totality of transformations $[\alpha_{ij}]$ in the $GF[p^n]$ defines a group $G_{p^{\mu n}}$. Indeed, we have $[\alpha_{ij}]\,[\beta_{ij}] = [\gamma_{ij}]$, where

$$(2) \qquad \gamma_{ij} = \beta_{ij} + \sum_{k=j+1}^{i-1} \beta_{ik}\alpha_{kj} + \alpha_{ij}$$
$$(i = 1, \cdots, m;\ j = 1, \cdots, i-1).$$

Within $GLH(m, p^n)$ every subgroup of order $p^{\mu n}$ is conjugate with $G_{p^{\mu n}}$.

3. *Powers of* $[\alpha_{ij}]$. — Let $[\alpha_{ij}]^r = [\rho_{ij}]$. In the matrix $[\rho_{ij}]$ the elements in a line parallel to the main diagonal are given by a single type of formula. We abbreviate the binomial coefficients as follows:

$$r_1 = r, \quad r_2 = \frac{r(r-1)}{1\cdot 2}, \quad r_3 = \frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}, \quad \text{etc.}$$

As shown by induction from r to $r + 1$, we have

$$\rho_{i\,i-1} = r_1\alpha_{i\,i-1}, \quad \rho_{i\,i-2} = r_1\alpha_{i\,i-2} + r_2\alpha_{i\,i-1}\alpha_{i-1\,i-2},$$

$$\rho_{i\,i-3} = r_1\alpha_{i\,i-3} + r_2(\alpha_{i\,i-2}\alpha_{i-2\,i-3} + \alpha_{i\,i-1}\alpha_{i-1\,i-3})$$
$$+ r_3\alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-3},$$

$$\rho_{i\,i-4} = r_1\alpha_{i\,i-4} + r_2(\alpha_{i\,i-3}\alpha_{i-3\,i-4} + \alpha_{i\,i-2}\alpha_{i-2\,i-4} + \alpha_{i\,i-1}\alpha_{i-1\,i-4})$$
$$+ r_3(\alpha_{i\,i-2}\alpha_{i-2\,i-3}\alpha_{i-3\,i-4} + \alpha_{i\,i-1}\alpha_{i-1\,i-3}\alpha_{i-3\,i-4}$$
$$+ \alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-4}) + r_4\alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-3}\alpha_{i-3\,i-4}.$$

For the general element $\rho_{i\,i-j}$, we have a sum of expressions, that with the factor r_k being a homogeneous function of the kth degree of the α's composed of all possible terms of the form

$$\alpha_{ia_1}\alpha_{a_1a_2}\alpha_{a_2a_3}\cdots\alpha_{a_{k-1}i-j} \qquad (i>a_1>a_2>\cdots>a_{k-1}>i-j).$$

For example, if $i=6, j=5$, we have

$$\rho_{61} = r_1\alpha_{61} + r_2(\alpha_{62}\alpha_{21} + \alpha_{63}\alpha_{31} + \alpha_{64}\alpha_{41} + \alpha_{65}\alpha_{51})$$
$$+ r_3(\alpha_{63}\alpha_{32}\alpha_{21} + \alpha_{64}\alpha_{42}\alpha_{21} + \alpha_{64}\alpha_{43}\alpha_{31} + \alpha_{65}\alpha_{52}\alpha_{21} + \alpha_{65}\alpha_{53}\alpha_{31}$$
$$+ \alpha_{65}\alpha_{54}\alpha_{41}) + r_4(\alpha_{64}\alpha_{43}\alpha_{32}\alpha_{21} + \alpha_{65}\alpha_{53}\alpha_{32}\alpha_{21} + \alpha_{65}\alpha_{54}\alpha_{42}\alpha_{21}$$
$$+ \alpha_{65}\alpha_{54}\alpha_{43}\alpha_{31}) + r_5\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32}\alpha_{21}.$$

4. *Period of* $[\alpha_{ij}]$. — Sufficient conditions for period r are

$$r_1 \equiv 0, \quad r_2 \equiv 0, \cdots, r_{m-1} \equiv 0 \pmod{p}.$$

Unless certain relations hold between the α_{ij}, these are also necessary conditions. *Hence* * *the period of* $[\alpha_{ij}]$ *is a divisor of* p^{q+1}, *where* p^q *is the highest power of* p *inferior to* m.

Corollary. For $p \equiv m$, $GLH(m, p^n)$ contains no operator of period p^2. An abstract group with operators of period $p^a(a>1)$ can not be represented as a linear homogeneous or linear fractional group in the $GF[p^n]$ on fewer than $p+1$ variables.

5. THEOREM. *The only self-conjugate transformations of* $G_{p^{\mu n}}$ *are*

$$\xi_i' = \xi_i, \quad \xi_m' = \xi_m + a_{m1}\xi_1, \qquad (i=1,\cdots,m-1). \tag{3}$$

* Jordan, Traité, p. 127 (r and m for his λ, ρ).

Let α_{ij} be fixed marks and β_{ij} be arbitrary marks. The conditions under which (2) remains unaltered upon the interchange of α_{ij} and β_{ij} are

$$\beta_{ij+1}\alpha_{j+1j} + \cdots + \beta_{ii-1}\alpha_{i-1j} = \alpha_{ij+1}\beta_{j+1j} + \cdots + \alpha_{ii-1}\beta_{i-1j}.$$

For $j < i-1$, no β enters more than once. Hence

$$\alpha_{j+1j} = 0, \cdots, \alpha_{i-1j} = 0\,;\ \alpha_{ij+1} = 0, \cdots, \alpha_{ii-1} = 0,$$
$$(i = 1, \cdots, m\,;\ j = 1, \cdots i-2).$$

The theorem being evident for $m \leqq 2$, let $m \geqq 3$. By the second set, for $i > 2, j = 1$,

$$\alpha_{i2} = \alpha_{i3} = \cdots = \alpha_{ii-1} = 0, \quad (i = 3, \cdots, m).$$

By the first set, for $i = m, j = 1$, $\alpha_{21} = \alpha_{31} = \cdots = \alpha_{m-11} = 0$.

6. THEOREM.* *The group $G_{p^{\mu n}}$ is transformed into itself by exactly $(p^n-1)^m p^{\mu n}$ transformations of $GLH(m, p^n)$, by $(p^n-1)^{m-1}p^{\mu n}$ transformations of $SLH(m, p^n)$, and by $1/d(p^n-1)^{m-1}p^{\mu n}$ transformations of $LF(m, p^n)$.*

Let $[A_{ij}]$ be an arbitrary transformation of $G_{p^{\mu n}}$, $[\alpha_{ij}]$ one to be determined. Now $[A_{ij}](\delta_{ij})$ and $(\delta_{ij})[\alpha_{ij}]$ replace ξ_i by

$$\sum_{j=1}^{m}(\delta_{ij} + \delta_{ij+1}A_{j+1j} + \delta_{ij+2}A_{j+2j} + \cdots + \delta_{im}A_{mj})\xi_j,$$

$$\sum_{j=1}^{m}(\delta_{ij} + \alpha_{ii-1}\delta_{i-1j} + \cdots + \alpha_{i3}\delta_{3j} + \alpha_{i2}\delta_{2j} + \alpha_{i1}\delta_{1j})\xi_j,$$

respectively. These are to be identical for arbitrary ξ's and A's. For $i = 1$, this requires

$$\delta_{12}A_{21} + \cdots + \delta_{1m}A_{m1} = 0, \quad \delta_{13}A_{32} + \cdots + \delta_{1m}A_{m2} = 0,$$
$$\delta_{14}A_{43} + \cdots + \delta_{1m}A_{m3}, \quad \cdots, \quad \delta_{1m}A_{mm-1} = 0.$$

whence $\delta_{1m} = 0, \cdots, \delta_{12} = 0$. For $i = 2$, the conditions are

$$\delta_{22}A_{21} + \cdots + \delta_{2m}A_{m1} = \delta_{11}\alpha_{21}, \quad \delta_{23}A_{32} + \cdots + \delta_{2m}A_{m2} = \delta_{12}\alpha_{21},$$

* For $m = 3$, the result follows more readily from § 5.

$$\delta_{24}A_{43} + \cdots + \delta_{2m}A_{m3} = \delta_{13}\alpha_{21}, \quad \cdots,$$

$$\delta_{2m}A_{m\,m-1} = \delta_{1\,m-1}\alpha_{21}, \quad 0 = \delta_{1m}\alpha_{21},$$

whence $\delta_{2m} = 0, \cdots, \delta_{24} = 0, \delta_{23} = 0, \delta_{22}A_{21} = \delta_{11}\alpha_{21}$. Relying upon induction, we suppose that after $i - 1$ such steps we have

$$\delta_{12} = \cdots = \delta_{1m} = 0, \ \delta_{23} = \cdots = \delta_{2m} = 0, \cdots,$$

$$\delta_{i-1\,i} = \cdots = \delta_{i-1\,m} = 0.$$

Then, in the ith step, the conditions, for $j = i, i+1, \cdots, m-1$, are

$$\delta_{i\,i+1}A_{i+1\,i} + \cdots + \delta_{im}A_{mi} = 0, \ \cdots, \ \delta_{im}A_{m\,m-1} = 0.$$

Hence $\delta_{im} = 0, \cdots, \delta_{ii+1} = 0$; the conditions, for $j = 1, \cdots, i - 1$, then serving to determine certain of the α_{ij}. It follows that *

$$(4) \qquad (\delta_{ij}) \equiv \begin{vmatrix} \delta_{11} & 0 & 0 & 0 & \cdots & 0 \\ \delta_{21} & \delta_{22} & 0 & 0 & \cdots & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

7. The inverse of $[\beta_{ij}]$ is found to be $[\beta'_{ij}]$, where

$$\beta'_{ii-1} = -\beta_{ii-1}, \ \beta'_{i\,i-2} = -\beta_{i\,i-2} + \beta_{ii-1}\beta_{i-1\,i-2},$$

$$\beta'_{ii-3} = -\beta_{i\,i-3} + \beta_{i\,i-2}\beta_{i-2\,i-3} + \beta_{ii-1}\beta_{i-1\,i-3}$$

$$- \beta_{i\,i-1}\beta_{i-1\,i-2}\beta_{i-2\,i-3}, \cdots$$

the literal parts following the law of § 3, while the coefficient is ± 1 according as the number of factors β is even or odd.

We now get $[\beta_{ij}]^{-1}\,[a_{ij}]\,[\beta_{ij}] = [\alpha'_{ij}]$, where

$$\alpha'_{ii-1} = \alpha_{ii-1}, \ \alpha'_{i\,i-2} = \alpha_{i\,i-2} + \beta_{ii-1}\alpha_{i-1\,i-2} - \alpha_{ii-1}\beta_{i-1\,i-2},$$

$$\alpha'_{ii-3} = \alpha_{i\,i-3} + \beta_{ii-1}\alpha_{i-1\,i-3} - \alpha_{ii-1}\beta_{i-1\,i-3} + \beta_{i\,i-2}\alpha_{i-2\,i-3}$$

$$- \alpha_{i\,i-2}\beta_{i-2\,i-3} + \alpha_{ii-1}\beta_{i-1\,i-2}\beta_{i-2\,i-3} - \beta_{ii-1}\alpha_{i-1\,i-2}\beta_{i-2\,i-3}.$$

* That the result is valid also for the linear fractional group may be shown by allowing the entrance of a factor Θ, where $\Theta^m = 1$ (cf. the author's Linear Groups, p. 242). That $\Theta = 1$ follows most readily by the theory of canonical forms.

In general, α'_{ii-j} equals α_{ii-j} together with a sum of binomials, each the difference of two terms involving the same subscripts, the two terms being interchanged upon interchanging the α and β with highest first subscript. The subscripts entering the various binomials follow the law of § 3. For example, we have

$$\alpha'_{51} = \alpha_{51} + \beta_{54}\alpha_{41} - \alpha_{54}\beta_{41} + \beta_{53}\alpha_{31} - \alpha_{53}\beta_{31} + \beta_{52}\alpha_{21} - \alpha_{52}\beta_{21}$$
$$+ \alpha_{53}\beta_{32}\beta_{21} - \beta_{53}\alpha_{32}\beta_{21} + \alpha_{54}\beta_{42}\beta_{21} - \beta_{54}\alpha_{42}\beta_{21} + \alpha_{54}\beta_{43}\beta_{31}$$
$$- \beta_{54}\alpha_{43}\beta_{31} + \beta_{54}\alpha_{43}\beta_{32}\beta_{21} - \alpha_{54}\beta_{43}\beta_{32}\beta_{21}.$$

8. *Conjugate operators of* $G_{p^{\mu}\nu_{\phi}}$. — Consider two operators $[\alpha_{ij}]$ and $[\alpha'_{ij}]$, with $\alpha'_{ii-1} = \alpha_{ii-1} (i = 2, \cdots, m)$. For $m = 3$ the further condition for conjugacy (§ 7) is $\alpha'_{31} = \alpha_{31} + \alpha_{21}\beta_{32} - \alpha_{32}\beta_{21}$. If α_{21} and α_{32} are not both zero, this condition may be satisfied by choice of β_{32}, β_{21}.

THEOREM. *For* $m = 3$, *the operators of* $G_{p^{2n}}$, *other than the* p^n *self-conjugate operators* (3), *fall into* $p^{2n} - 1$ *distinct sets* $S_{\alpha_{21}, \alpha_{32}}$ *of conjugates* $[\alpha_{ij}]$, *with* α_{21} *and* α_{32} *fixed marks not both zero and* α_{31} *ranging over the field.*

For $m > 3$, we set $\alpha'_{ij} - \alpha_{ij} = \epsilon_{ij}$ for $j < i - 1$. Then, for $m = 4$, the further conditions for conjugacy (§ 7) are

$$(5) \qquad \epsilon_{31} = \beta_{32}\alpha_{21} - \alpha_{32}\beta_{21}, \qquad \epsilon_{42} = \beta_{43}\alpha_{32} - \alpha_{43}\beta_{32},$$

$$(6) \qquad \epsilon_{41} = \beta_{43}\alpha_{31} - \alpha_{43}\beta_{31} + \beta_{42}\alpha_{21} - \alpha_{42}\beta_{21} + \alpha_{43}\beta_{32}\beta_{21} - \beta_{43}\alpha_{32}\beta_{21}.$$

(*a*) Let $\alpha_{32} \neq 0$. Then β_{21} and β_{43} are determined by (5). In case $\alpha_{43} \neq 0$, β_{31} is determined by (6). For $\alpha_{43} = 0$, (6) becomes, by (5),

$$\beta_{42}\alpha_{32}\alpha_{21} - \beta_{32}\alpha_{21}\alpha'_{42} + \alpha'_{31}\epsilon_{42} - \alpha_{32}\epsilon_{41} + \alpha_{42}\epsilon_{31} = 0.$$

This determines β_{42} if $\alpha_{21} \neq 0$. For $\alpha_{21} = 0$, it may be written

$$(7) \qquad \alpha'_{31}\alpha'_{42} - \alpha'_{32}\alpha'_{41} = \alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41}.$$

(*b*) Let $\alpha_{32} = 0$. Conditions (5) require $\alpha_{21}\epsilon_{42} + \alpha_{43}\epsilon_{31} = 0$, viz.,

$$\alpha'_{42}\alpha'_{21} + \alpha'_{43}\alpha'_{31} = \alpha_{42}\alpha_{21} + \alpha_{43}\alpha_{31}. \tag{8}$$

Assume that this condition is satisfied. If $\alpha_{21} \neq 0$, (5) and (6) determine β_{32} and β_{42}; if $\alpha_{21} = 0$, $\alpha_{43} \neq 0$, they determine β_{32} and β_{31}. If $\alpha_{21} = \alpha_{43} = 0$, then

$$\epsilon_{31} = 0, \qquad \epsilon_{42} = 0, \qquad \epsilon_{41} = \beta_{43}\alpha_{31} - \alpha_{42}\beta_{21}.$$

The latter may be satisfied by choice of β_{43} and β_{21} unless $\alpha_{31} = \alpha_{42} = 0$.

For $m = 4$, two transformations $[\alpha_{ij}]$ and $[\alpha'_{ij}]$ with the same α_{21}, α_{32}, α_{43}, are conjugate (a) if $\alpha_{32} \neq 0$, provided (7) holds when $\alpha_{21} = \alpha_{43} = 0$; (b) if $\alpha_{32} = 0$ with (8) satisfied, provided $\alpha'_{31} = \alpha_{31}$ and $\alpha'_{42} = \alpha_{42}$ when $\alpha_{21} = \alpha_{43} = 0$, while also $\alpha'_{41} = \alpha_{41}$ when also $\alpha_{31} = \alpha_{42} = 0$.

THEOREM. *For $m = 4$, the operators of $G_{p^{6n}}$, other than the p^n self-conjugate operators (3) fall into distinct sets of conjugates as follows:*

$(p^{2n} - 1)(p^n - 1)$ *sets* $S_{\alpha_{21},\alpha_{32},\alpha_{43}}$, $\alpha_{32} \neq 0$, α_{21} *and* α_{43} *not both zero;*

$p^n(p^n - 1)$ *sets* $S_{\alpha_{32},\tau}$, $\alpha_{32} \neq 0$, $\alpha_{21} = \alpha_{43} = 0$, $\tau = \alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41}$;

$p^n(p^{2n} - 1)$ *sets* $\sum_{\alpha_{21},\alpha_{43},\sigma}$, $\alpha_{32} = 0$, α_{21} *and* α_{43} *not both zero,* $\sigma = \alpha_{42}\alpha_{21} + \alpha_{43}\alpha_{31}$;

$p^{2n} - 1$ *sets* $\sum_{\alpha_{21},\alpha_{43}}$, $\alpha_{21} = \alpha_{32} = \alpha_{43} = 0$, α_{31} *and* α_{42} *not both zero;*

each set of the four types containing respectively p^{3n}, p^{2n}, p^{2n}, p^n conjugate operators.

By way of check, we note that

$$p^n + (p^{2n} - 1)(p^n - 1)p^{3n} + p^n(p^n - 1)p^{2n} + p^n(p^{2n} - 1)p^{2n} + (p^{2n} - 1)p^n = p^{6n}.$$

9. *Self-conjugate subgroups of $G_{p^{3n}}$ for $m = 3$, $n = 1$.** — The only ones are readily seen, in view of the first theorem

* The method holds for general n. Cf. *Transactions*, vol. 4 (1903), p. 376.

of § 8, to be G_{p^3} itself, identity, the cyclic group C_p of the self-conjugate operators (3) and the $p+1$ groups *.

$$(9) \qquad K_{p^2}: \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{vmatrix}, \quad H^t_{p^2}: \begin{vmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ b & rt & 1 \end{vmatrix},$$

where $b, c, r, t = 0, 1, \cdots, p-1$, t constant for each H.

Proceeding as in § 6, we find that a ternary transformation (a_{ij}) transforms K_{p^2} into itself if and only if $\alpha_{13} = \alpha_{23} = 0$. For $H^0_{p^2}$ the conditions are $\alpha_{12} = \alpha_{13} = 0$. For $H^t_{p^2}$, $t \neq 0$, the conditions are $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$, $\alpha^2_{22} = \alpha_{11}\alpha_{33}$.

THEOREM. *Each of the groups K_{p^2} and $H^0_{p^2}$ is transformed into itself by exactly $(p^2-1)(p-1)^2p^3$ transformations of GLH (3, p), by $(p^2-1)(p-1)p^3$ transformations of SLH (3, p) and by $(p^2-1)(p-1)p^3/d$ transformations of LF(3, p). For each group $H^t_{p^2}$ $(t \neq 0)$, the numbers are respectively*

$$(p-1)^2p^3, \quad d(p-1)p^3, \quad (p-1)p^3.$$

10. *Self-conjugate subgroups H of G_{p^m} for $m = 4$, $n = 1$.* — We make use of the final theorem of § 8. If H contains one operator of the set $S_{\alpha_{21}, \alpha_{32}, \alpha_{43}}$, it necessarily contains the group †

$$(10) \qquad H^{rs}_{p^4}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ rt & 1 & 0 & 0 \\ a & r & 1 & 0 \\ b & c & rs & 1 \end{vmatrix}$$

where t and s are fixed integers not both zero.

Consider next the set $S_{\alpha_{32}, \tau}$. Denote by $\{\alpha\}$ the general operator $[\alpha_{ij}]$ with $\alpha_{21} = \alpha_{43} = 0$. Then

* Note that $\xi'_3 = t^{-1}\xi_3$ transforms $H^t_{p^2}$ into $H^1_{p^2}$. For SLH $(3, p)$ and LF $(3, p)$, we transform $[r, B, rt]$ by $\xi'_1 = \tau\xi_1$, $\xi'_2 = \tau^{-1}\xi_2$, $\xi'_3 = \xi_3$, and obtain $[r', \tau^{-1}B, r't']$, where $r' = \tau^{-2}r$, $t' = \tau^3 t$. We therefore restrict t to one of d values (see § 2).

† Henceforth, in the definition of a group, all letters appearing in the matrix of its general transformation, aside from the constants given also as superscripts in the symbol for the group, take independently the values $0, 1, \cdots, p-1$.

$$\{\alpha\}\{\beta\} = \{\alpha + \beta\}, \quad \{\alpha\}^r = \{r\alpha\}.$$

If $\{\alpha\}$ and $\{\beta\}$ belong to the same (present) set while $r - r'$ and $R - R'$ are not both zero, then $\{r\alpha + R\beta\} = \{r'\alpha + R'\beta\}$ is seen to require $\beta_{31} = \alpha_{31}$, $\beta_{41} = \alpha_{41}$, $\beta_{42} = \alpha_{42}$. Hence the operators $\{r\alpha + R\beta\}$; r, $R = 0, 1, \cdots, p-1$ are all distinct when $\{\alpha\}$ and $\{\beta\}$ are. If $\{\alpha\}$, $\{\alpha'\}$, $\{\beta\}$ have the same values for τ and α_{32}, the conditions that $\{r\alpha + R\beta\}$ equals $\{r'\alpha' + R'\beta\}$ are

$$r + R = r' + R', \qquad r(\alpha_{31}\alpha_{42} - \beta_{31}\beta_{42}) = r'(\alpha'_{31}\alpha'_{42} - \beta_{31}\beta_{42}),$$

$$r(\alpha_{31} - \beta_{31}) = r'(\alpha'_{31} - \beta_{31}), \qquad r(\alpha_{42} - \beta_{42}) = r'(\alpha'_{42} - \beta_{42}).$$

Eliminating r and r' from the last three, we get

$$(\alpha_{42} - \beta_{42})\alpha'_{31} - (\alpha_{31} - \beta_{31})\alpha'_{42} + \alpha_{31}\beta_{42} - \alpha_{42}\beta_{31} = 0,$$

$$(\alpha_{31} - \beta_{31})\alpha'_{31}\alpha'_{42} - (\alpha_{31}\alpha_{42} - \beta_{31}\beta_{42})\alpha'_{31} + (\alpha_{42} - \beta_{42})\alpha_{31}\beta_{31} = 0.$$

Eliminating α'_{42} from these, we get

$$(\alpha_{42} - \beta_{42})(\alpha'_{31} - \alpha_{31})(\alpha'_{31} - \beta_{31}) = 0.$$

Let first $\alpha_{42} \neq \beta_{42}$. For $\alpha'_{31} = \alpha_{31}$, we get $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \alpha_{42}) = 0$, whence $\{\alpha'\} = \{\alpha\}$ if $\alpha_{31} \neq \beta_{31}$. For $\alpha'_{31} = \beta_{31}$, we get $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \beta_{42}) = 0$, whence $\{\alpha'\} = \{\beta\}$ if $\alpha_{31} \neq \beta_{31}$. Let next $\alpha_{42} \beta_{42}$. Then $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \alpha_{42}) = 0$, whence $\{\alpha\} = \{\beta\}$ *if* $\alpha_{31} = \beta_{31}$, while $\alpha'_{42} = \alpha_{42} - \beta_{42}$ if $\alpha_{31} \neq \beta_{31}$. *Hence if* $\{\alpha\}$, $\{\alpha'\}$ *and* $\{\beta\}$ *are distinct, and if* $R \neq R'$ *when* $r' = r = 0$, *then* $\{r\alpha + R\beta\} \neq \{r'\alpha' + R'\beta\}$ *except for* $\alpha_{31} = \beta_{31}$, $\alpha_{42} \neq \beta_{42}$ *and for* $\alpha'_{42} = \alpha_{42} = \beta_{42}$, $\alpha_{31} \neq \beta_{31}$.

It follows that, if the α_{ij} take all sets of values such that $\alpha_{31} \neq \beta_{31}$, $\alpha_{42} \neq \beta_{42}$, $\alpha_{32} =$ constant $\neq 0$, $\tau =$ constant, no two of the resulting sets of operators $\{r\alpha + R\beta\}$ have in common an operator other than the $\{R\beta\}$. Now $\alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41} = \tau$ has $(p-1)^2$ sets of solutions modulo p, since α_{31} has any value $\neq \beta_{31}, \alpha_{42}$ any value $\pm \beta_{42}$, whence α_{41} is uniquely determined. But $(p-1)^2(p^2-p) > p^3$ if $p^2(p-5) + p(p+2) + p - 1 > 0$, viz., if $p > 3$. For $p = 3$, these $(p-1)^2(p^2-p)$ operators, the p operators $\{R\beta\}$, and any one of the operators with $\alpha_{31} = \beta_{31}, \alpha_{42} \neq \beta_{42}$, give more than p^3 distinct operators. For $p = 2$, the 4 operators of the set $S_{1,0}$ are commutative and of

period 2, while 6 of their products two at a time furnish 6 distinct operators of the set $\sum_{a_{31}, a_{42}}$. Likewise the 4 operators of set $S_{1,1}$ furnish at once 6 further operators. *Hence for any p, if H contains an operator of the set $S_{a_{32}, \tau}$, it contains the group $H_{p^3}^{0,0}$.*

Next, let H contain an operator of the set $\sum_{a_{21}, a_{43}, \sigma}$. Denote by $[\alpha]'$ an operator $[\alpha]$ having $\alpha_{32} = 0$. Then $[\alpha]'[\beta]' = [\delta]'$, where

$$\delta_{41} = \alpha_{41} + \beta_{41} + \alpha_{31}\beta_{43} + \alpha_{21}\beta_{42}, \quad \delta_{ij} = \alpha_{ij} + \beta_{ij}, \ (i, j) \neq (4, 1).$$

The inverse of $[\alpha]'$ is $[\gamma]'$, where $\gamma_{ij} = -\alpha_{ij}$, $(i, j) \neq (4, 1)$, while $\gamma_{41} = -\alpha_{41} + \sigma$. If $[\alpha]'$ and $[\beta]'$ are conjugate, so that

$$\alpha_{21} = \beta_{21}, \ \alpha_{43} = \beta_{43}, \ \alpha_{21}(\alpha_{42} - \beta_{42}) + \alpha_{43}(\alpha_{31} - \beta_{31}) = 0,$$

it therefore follows that

$$(11) \quad [\alpha]'[\beta]'^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_{31} - \beta_{31} & 0 & 1 & 0 \\ \alpha_{41} - \beta_{41} + \alpha_{21}\alpha_{42} - \alpha_{21}\beta_{42} & \alpha_{42} - \beta_{42} & 0 & 1 \end{vmatrix}.$$

Varying the α_{ij} but so as to preserve the invariant σ, we conclude that H contains p^2 operators of the form (11), including the p self-conjugate operators (3). Hence H is of order $\equiv p^3$. Now $[\alpha]'^r$ equals $[\epsilon]'$, where $\epsilon_{ij} = r\alpha_{ij}(i, j) \neq (4, 1)$, while $\epsilon_{41} = r\alpha_{41} + 1/2r(r-1)\sigma$. Consider the product of $[\alpha]'^r$ by (11). According as $\alpha_{21} \neq 0$ or $\alpha_{21} = 0$, $\alpha_{43} \neq 0$, we obtain, after a simplification of the notations, the groups

$$H_{p^3}^{l,t}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ b & rl - ta & tr & 1 \end{vmatrix}, \qquad K_{p^3}^{s}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ sr & 0 & 1 & 0 \\ b & c & r & 1 \end{vmatrix}.$$

Finally, if H contains an operator of the set $\sum_{a_{31}, a_{42}}$, we find immediately that it contains one of the commutative groups:

$$K_{p^3}^t: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ b & rt & 0 & 1 \end{vmatrix}, \quad G_{p^3}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & r & 0 & 1 \end{vmatrix},$$

where $b, r = 0, 1, \cdots, p-1$ in each, while t is any fixed integer.

The next step consists in determining all the distinct groups generated by two or more of the preceding. The product of the general operator of $H_{p^4}^{t,s}$ by that, written in capitals, of $H_{p^4}^{T,S}$ is of the form (α_{ij}) with

$$\alpha_{21} = RT + rt, \ \alpha_{32} = R + r, \ \alpha_{43} = RS + rs.$$

Eliminating R and r, we get

$$(s - S)\alpha_{21} - (t - T)\alpha_{43} + (tS - sT)\alpha_{32} = 0.$$

According as $s - S \neq 0$ or $s - S = 0$, $t - T \neq 0$, we obtain

$$H_{p^3}^{v,w}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ vg + w\rho & 1 & 0 & 0 \\ a & \rho & 1 & 0 \\ b & c & g & 1 \end{vmatrix}, \quad H_{p^3}^{s}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ e & 1 & 0 & 0 \\ a & \rho & 1 & 0 \\ b & c & s\rho & 1 \end{vmatrix}.$$

The product of the general operator of $H_{p^3}^{l,t}$ by that, written in capital letters, of $H_{p^3}^{L,T}$ is of the form (α_{ij}) with

$$\alpha_{21} = R + r, \ \alpha_{31} = A + a, \ \alpha_{42} = RL + rl - TA - ta,$$

$$\alpha_{43} = RT + rt.$$

The determinant of the coefficients of R, r, A, a equals $(t - T)^2$. According as $t - T \neq 0$ or $t - T = 0$, $l - L \neq 0$, we get

$$G_{p^5}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & l & 1 \end{vmatrix}, \quad G_{p^4}^{t}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & t\rho & 1 \end{vmatrix}.$$

For $s \neq t$, (K^s_{p3}, K^t_{p3}) and (K^s_{p2}, K^t_{p2}) give, respectively,

$$L_{p4}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & \rho & 1 \end{vmatrix}, \quad G_{p3}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & 0 & 1 \end{vmatrix}.$$

We obtain immediately the further results

$$(K^t_{p3}, G_{p3}) = G_{p3}, \qquad (H^{t,t}_{p3}, G_{p3}) = G^t_{p4},$$
$$(H^{\tau,s}_{p4}, H^{t,0}_{p3}) = H^t_{p3}, \qquad (H^{\tau,s}_{p4}, H^{t,\tau}_{p3}) = H^{t^{-1},\, \tau - t^{-1}s}_{p3},$$
$$(H^{t,\tau}_{p3}, K^s_{p3}) = G_{p3}, \qquad (H^{t,\tau}_{p3}, K^t_{p3}) = G^\tau_{p4},$$
$$(H^{t,s}_{p4}, K^s_{p3}) = H^{0,t}_{p3}, \qquad (H^{t,s}_{p4}, L_{p4}) = H^{0,t}_{p3},$$
$$(H^{t,s}_{p4}, G^0_{p4}) = H^s_{p3}, \qquad (H^{t,s}_{p4}, G^\tau_{p4}) = H^{\tau^{-1},\, t-\tau^{-1}s}_{p3},$$
$$(H^{t,t}_{p3}, L_{p4}) = G_{p3}, \qquad (H^{t,t}_{p3}, G^\tau_{p4}) = G_{p3} (T \neq t),$$
$$(H^{t,t}_{p3}, G_{p3}) = L_{p4}, \qquad (K^s_{p3}, K_{p3}) = L_{p4},$$
$$(K^s_{p3}, G_{p3}) = L_{p4}, \qquad (K^s_{p3}, G^t_{p4}) = G_{p3}.$$

For the remaining pairs, either one group is a subgroup of the other, or else one is of order p^5 and the two generate G_{p6} itself.

THEOREM. *The only self-conjugate subgroups of G_{p6} other than itself and identity are $p^2 + p + 1$ groups $H^{v,w}_{p5}$, H^s_{p5}, G_{p5} of order p^5; $p^2 + p + 1$ groups $H^{t,s}_{p4}$, G^t_{p4}, L_{p4} of order p^4; $p^2 + p + 1$ groups $H^{t,t}_{p3}$, K^s_{p3}, G_{p3} of order p^3; $p + 1$ groups K^t_{p2}, G_{p2} of order p^2; and a single group C_p of order p, formed of the self-conjugate operators* (3).

11. To determine all quaternary transformations (α_{ij}) which transform $H^{v,w}_{p5}$ into itself, let t denote an arbitrary one of its operators and T (its elements denoted by capital letters) one to be determined so that $(\alpha_{ij})T = t(\alpha_{ij})$. For v and w not both zero, the conditions are

$$\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \alpha_{34} = 0, \quad G\alpha_{33} = g\alpha_{44}, \quad R\alpha_{22} = \rho\alpha_{33},$$
$$C\alpha_{22} + G\alpha_{32} = \rho\alpha_{43} + c\alpha_{44}, \quad A\alpha_{11} + R\alpha_{21} = (vg + w\rho)\alpha_{32} + a\alpha_{33},$$

$$B\alpha_{11} + C\alpha_{21} + G\alpha_{31} = (vg + w\rho)\alpha_{42} + a\alpha_{43} + b\alpha_{44},$$

$$(vG + wR)\alpha_{11} = (vg + w\rho)\alpha_{22}.$$

Hence G, R, C, A, B are uniquely determined. The final condition gives

$$v\alpha_{11}\alpha_{44}\alpha_{33}^{-1} = v\alpha_{22}, \quad w\alpha_{11}\alpha_{33}\alpha_{22}^{-1} = w\alpha_{22}.$$

For $v = 0$, $w \neq 0$, $H_{p^s}^{v,w}$ is transformed into itself by respectively

$$(12) \qquad (p-1)^3p^6, \quad (p-1)^2p^6, \quad (p-1)^2p^6/d$$

transformations of $GLH(4, p)$, $SLH(4, p)$, $LF(4, p)$. For $v \neq 0$, $w = 0$, and for $v \neq 0$, $w \neq 0$, the numbers are respectively

$$(p-1)^3p^6, \ a(p-1)^2p^6, \ \frac{a}{d}(p-1)^2p^6; \qquad \begin{pmatrix} a = 2, \text{ if } p > 2 \\ a = 1, \text{ if } p = 2 \end{pmatrix}.$$

$$(p-1)^2p^6, \ a(p-1)p^6, \ \frac{a}{d}(p-1)p^6.$$

For $H_{p^s}^{0,0}$, the conditions on (α_{ij}) are that α_{13}, α_{14}, α_{23}, α_{24}, α_{34} all vanish. For $H_{p^s}^{0}$, the conditions are that α_{12}, α_{13}, α_{14}, α_{23}, α_{24} all vanish; for $H_{p^s}^{s}$, $s \neq 0$, the additional conditions are $\alpha_{34} = 0$, $\alpha_{33}^2 = \alpha_{22}\alpha_{44}$. For G_{p^s}, the conditions are that α_{12}, α_{13}, α_{14}, α_{24}, α_{34} all vanish.

Each of the groups $H_{p^s}^{0,0}$, $H_{p^s}^{0}$, G_{p^s} is transformed into itself by exactly $(p^2-1)(p-1)^3p^6$ transformations of $GLH(4, p)$, by $(p^2-1)(p-1)^2p^6$ of $SLH(4, p)$, by $1/d(p^2-1)(p-1)^2p^6$ of $LF(4, p)$. For $H_{p^s}^{s}$, $s \neq 0$, the corresponding numbers are given by (12).

THE UNIVERSITY OF CHICAGO,
February 6, 1904.

THE EXTERIOR AND INTERIOR OF A PLANE CURVE.*

BY DR. G. A. BLISS.

THE theorem that a continuous closed curve without double points divides the plane into two regions, an exterior and an interior, has been discussed by several writers. Jordan † and Schoenflies ‡ assume the theorem for polygons and then extend it to apply to more general curves by processes which are extremely complicated. In the March number of the BULLETIN, Ames has sketched an apparently simpler method which applies to the so-called *regular* curves. The writer ventures to give below another proof for a class of curves which is more general in that the tangent is not assumed to exist, but which on the other hand may not include all of the regular curves. Polygons, curves consisting of a finite number of analytic pieces, and *regular* curves of the type treated by Schoenflies, are among those to which the method applies.

§ 1. *Hypotheses on the Curve.*

The curve C is taken in the form

$$x = \phi(t), \quad y = \psi(t), \tag{1}$$

where ϕ, ψ are considered for values of t in the interval

$$t_0 \leqq t \leqq T \tag{T}$$

and are supposed to have the following properties:

1) ϕ, ψ are continuous on $[T]$;

2) the curve is closed, *i. e.*,

$$(\phi(t_0), \psi(t_0)) = (\phi(T), \psi(T)),$$

but otherwise has no multiple points;

3) the function ϕ is an increasing § or a decreasing function for all but a finite number of points of $[T]$.

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* Read before the Mathematical Club of the University of Chicago, October 23, 1903.

† Cours d'Analyse, vol. 1, pp. 91–99.

‡ *Göttinger Nachrichten*, 1896, p. 85.

§ A function is said to be increasing at a point t' when an interval $[t' - \delta, t' + \delta]$ can be found such that

The parameter representation can be so arranged that the t values mentioned in 3), the maxima and minima of ϕ, are

$$t_0, t_1, t_2, \cdots, t_n = T,$$

where $t_{k-1} < t_k$, $(k = 1, 2, \cdots, n)$, and t_0, t_n correspond to the smallest value of x on the curve.

Consider the interval $[t_{k-1}, t_k]$.* On account of 3) the function $x = \phi(t)$ is monotonic in this interval, and consequently the inverse function $t = t(x)$ is single-valued and continuous in the interval $[x_{k-1}, x_k]$, when x_{k-1} and x_k are the values of x corresponding to t_{k-1} and t_k respectively.

By substitution in (1),

$$C_k: \qquad y = \psi(t(x)) = \psi_k(x), \qquad [x_{k-1}, x_k],$$

which is also a single-valued continuous function of x in $[x_{k-1}, x_k]$.

The curve is therefore divided into n continuous pieces C_k $(k = 1, 2, \cdots, n)$. Any piece C_k has its endpoints (x_{k-1}, y_{k-1}), (x_k, y_k) in common with C_{k-1} and C_{k+1} respectively, but otherwise has no intersections with the other pieces.

§ 2. *Classification of the Points of the Plane by Means of a Continuous Function $g(x, y)$.*

Consider the piece C_1. The two pieces of horizontal lines, $y = y_0$ for $x \leqq x_0$, and $y = y_1$ for $x \geqq x_1$, adjoined to C_1 (Fig. 1), together define a continuous function

$$y = Y_1(x), \quad [-\infty < x < +\infty].$$

The equation

$$g_1(x, y) = y - Y_1(x)$$

$$\phi(t) < \phi(t') \quad \text{for } t' - \delta < t < t',$$
$$\phi(t) > \phi(t') \quad \text{for } t' < t < t' + \delta.$$

This third assumption is made also by Jordan, Cours d'Analyse, vol. 2, p. 132, in reducing to a line integral the integral

$$\iint \frac{\partial F}{\partial y}\, dxdy,$$

taken over a region bounded by a curve.

* The interval in which a function is to be considered will always be indicated as here by a square bracket.

defines g_1 as a function of x and y, continuous over the whole plane. The product

$$g(x, y) = \prod_{k=1}^{n} g_k(x, y)$$

has the same property, where g_k $(k = 1, 2, \cdots, n)$ is supposed to have been formed from C_k in a manner analogous to the formation of g_1.

Near a point of C, g can take both positive and negative

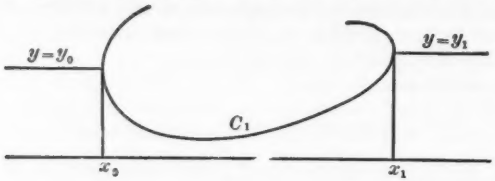

FIG. 1.

values. For if (ξ, η) is in C_k but not an end point, then for points on the line $x = \xi$, the function g has the form

$$g(\xi, y) = (y - \eta)^{2\mu+1}\overline{g}(\xi, y), \tag{2}$$

where $\overline{g} \neq 0$ in the neighborhood of $y = \eta$. This follows because the factor $g_k(\xi, y)$ has the form

$$g_k(\xi, y) = y - \eta.$$

If any other factor g_λ has the value $y - \eta$ for $x = \xi$, then either $g_{\lambda-1}$ or $g_{\lambda+1}$ has the same value. Similarly g_λ takes opposite signs near the end points of C_k because they are the limiting points of the interior points of the piece.

In the neighborhood of any point (ξ, η) not on C, g always takes values different from zero, but never takes values opposite in sign. For near such a point, g has the form

$$g(x, y) = (y - \eta)^{2\mu}\overline{g}(x, y), \tag{3}$$

where $\overline{g} \neq 0$ in the neighborhood of (ξ, η).

The points of the plane can now be classified into :

1) *points of C, near which g takes both positive and negative values;*

2) *interior points, near which g takes negative values but no positive ones;*

3) *exterior points, near which g takes positive values but no negative ones.*

If $i(\xi, \eta)$ is an interior (exterior) point, then it is always possible to find a circle about i in which all the points are interior (exterior). This follows from (3) if the circle is taken so small that $\bar{g} \neq 0$ in it, and furthermore of radius smaller than the shortest distance from i to the pieces C_k which give rise to factors in the form $y - \eta$. For in such a circle these factors all retain the same form, and g has consequently the sign of $\bar{g}(\xi, \eta)$ at every point of the circle where it does not vanish.

§3. *Curves Joining an Interior with an Exterior Point.*

Suppose a curve

$$D: \qquad x = f(\tau), \quad y = g(\tau),$$

starting at an exterior point for $\tau = \tau_0$, and attaining an interior point for $\tau = \tau_1$ $(\tau_0 < \tau_1)$. Let τ' be the upper bound of the τ values such that the interval $[\tau_0, \tau]$ defines only exterior points on D. τ' can not correspond to an interior (exterior) point (x', y'), for then all points in a certain circle about (x', y') would be interior (exterior) points, and consequently τ' would be too large (too small) to be the upper limit in question. It must therefore define a point of the curve C.

If a continuous curve joins an exterior point with an interior point, it must have on it at least one point of C

§ 4. *Construction of I-Curves and E-Curves Parallel to C.*

For purposes of construction two positive quantities δ and ϵ can be selected so that for $k = 1, 2, 3, \cdots, n$,

1) $\delta < \frac{1}{2} \mid x_k - x_{k-1} \mid$;

2_a) $\epsilon <$ smallest distance from C_k to any non-adjacent piece;

2_b) $2\epsilon <$ smallest distance from C_k on $[x_{k-1}, x_k + (-1)^k\delta]$ to C_{k+1}.

By means of the curve pieces

$$C'_k: \qquad y = x_k(x) + (-1)^k\epsilon, \qquad [x_{k-1}, x_k],$$

a closed continuous curve can be constructed which does not intersect C. Consider for example C'_1 and C'_2. On account of 2_a) the only pieces of C which C'_1 can intersect are C_n and

C_2, and from 2_b) it can intersect C_2 only on $[x_1 - \delta, x_1]$. If

$$\psi_2(x) - \psi_1(x) > 0$$

for all points common to $[x_0, x_1]$ and $[x_1, x_2]$ (see Fig. 2), then C_1' and C_2' have no points of intersection, but their ends on the

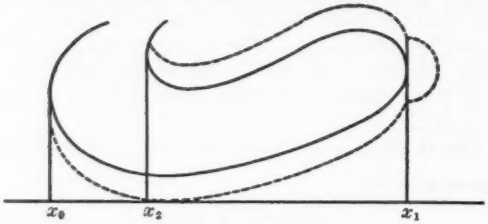

FIG. 2.

ordinate $x = x_1$ can be joined by a semicircle in the half-plane $x \geqq x_1$ with radius ϵ and center (x_1, y_1). The semicircle has no points in common with C on account of 2_a), and the resulting

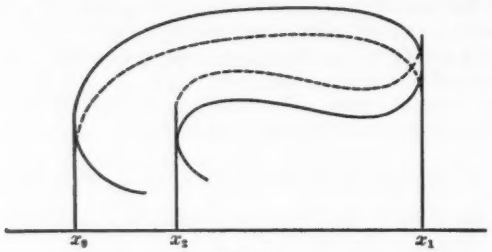

FIG. 3.

continuous curve formed by joining C_1', C_2' in this manner does not intersect C_1 or C_2.

On the other hand if

$$\psi_2(x) - \psi_1(x) < 0$$

(Fig. 3), then C_1' and C_2' will intersect for some smallest value x' between $x_1 - \delta$ and x_1. For at $x = x_1 - \delta$

$$(\psi_2 + \epsilon) - (\psi_1 - \epsilon) = \psi_2 - \psi_1 + 2\epsilon < 0$$

on account of 2_b), and at $x = x_1$

$$(\psi_2 + \epsilon) - (\psi_1 - \epsilon) = +2\epsilon > 0.$$

If now the points of C_1' and C_2' between x' and x_1 are disregarded, then the resulting curve is continuous and does not intersect either C_1 or C_2.

Suppose the same process carried out for each pair C_k', C_{k+1}'. The result is a continuous closed curve D' which does not intersect C. For any C_k' can only intersect C_{k-1} or C_{k+1}, and it has been joined with C_{k-1}' so as to avoid intersection with C_{k-1}, and with C_{k+1}' so that it does not intersect C_{k+1}.

A similar curve D'' can be constructed out of the pieces

$$C_k'': \quad y = \psi_k(x) + (-1)^{k-1}\epsilon, \quad [x_{k-1}, \quad x_k + (-1)^k\delta].$$

It is easy to show that D' and D'' lie on opposite sides of C. For the two points $(x_0 + \delta, \psi_1(x_0 + \delta) \pm \epsilon)$ lie on D' and D'' respectively, and are furthermore on opposite sides of C, as one discovers from the behavior of $g(x, y)$ (see equation (2)).

The result is then that *for a given* δ, ϵ *satisfying* $1)$, $2_a)$, $2_b)$, *two curves parallel to* C *can be constructed, the one entirely interior* (*I-curve*), *the other entirely exterior* (*E-curve*) to C. It should be noticed that the part of D' due to C_k' is cut by every ordinate on $[x_{k-1} + (-1)^{k-1}\delta, x_k + (-1)^k\delta]$.

§ 5. *Curves Joining Two Interior* (*Exterior*) *Points Without Intersecting the Curve C.*

Consider first a single interior point i (ξ, η). On $x = \xi$ there are at most a finite number of points of C because the number of pieces C_k is finite. There must be one at least with ordinate $> \eta$, for otherwise i could be joined to a point for which $g > 0$ by the ordinate $x = \xi$. Suppose the nearest to (ξ, η) lies on C_k, and let δ and ϵ be again restricted as follows:

a) when $\xi = x_k$ or $\xi = x_{k-1}$ take δ as in 1), and ϵ satisfying

$$2_c) \qquad \epsilon < \psi_k(\xi) - \eta$$

besides $2_a)$, $2_b)$;

b) when ξ lies between x_{k-1} and x_k, take δ satisfying

$$1_a) \qquad \delta < |\xi - x_{k-1}| \quad \text{and} \quad < |\xi - x_k|$$

besides 1), and ϵ as in *a*).

In either case the point $(\xi, \psi_k(\xi) - \epsilon)$ lies on the I-curve corresponding to these values of ϵ and δ, on account of § 3 and the fact that no point of C lies on the segment of $x = \xi$ which joins (ξ, η) and $(\xi, \psi_k(\xi) - \epsilon)$.

Any exterior point can be joined by straight line segments not intersecting C with another point of the same kind $e(\xi, \eta)$, for which ξ lies between the maximum and minimum values of x on C. The construction given above with slight modification can then be applied in order tc join e with an E-curve.

In a similar manner *an I-curve (E-curve) can be found with which any two interior (exterior) points can be connected by means of straight line segments not intersecting C.*

§6. *Some Classes of Curves to Which the Preceding Results Apply.*

A curve consisting of analytic * pieces can be represented in the form described in § 1 by choosing for the y-axis a direction a which is not parallel to any straight line piece. The only points where x can have maxima or minima are the ends of the pieces and points where the tangent is parallel to the y-axis. The latter are finite in number. For on any piece not a straight line, the fraction $\phi' / \sqrt{(\phi'^2 + \psi'^2)}$ is everywhere analytic and not constant. It would take the value cos a an infinity of time only if it were constant and equal to cos a on the whole piece.

In a similar manner the Schoenflies curves will have the desired form provided that the y-axis is not parallel to one of the straight pieces. For on the other pieces the direction of the tangent is presupposed to vary monotonically.

By properly choosing the coördinate axes the curves of character described in § 1 *can be made to include all those consisting of a finite number of analytic pieces, all the curves considered by Schoenflies, and all the so-called regular curves for which there exists a direction parallel to only a finite number of tangents to the curve.*

THE UNIVERSITY OF CHICAGO,
October, 1903.

* *I. e.*, on each piece φ and ψ are given by continuations of a single pair of analytic functions. Furthermore $\varphi'^2 + \psi'^2$ is supposed $\neq 0$ except at perhaps a finite number of points, which can be made the end-points of pieces.

RICCATI ISOTHERMAL SYSTEMS — A CORRECTION.

BY DR. EDWARD KASNER.

The list of solutions given on page 346 of my paper in the April BULLETIN is incomplete.* The error appears in the last step of the discussion of case III, where it is stated that the coefficient α must vanish.

When the value †

$$P = \alpha x + \gamma x^{-1} + \delta x^{-2} + \beta^2 \alpha^{-1} x^{-3}$$

is substituted in (13), or, more conveniently, in the equation

$$(2\alpha x^{-1} + P')(1 + P^2) - 2P(P'^2 + \beta^2 x^{-4}) = 0$$

derived by putting $y = 0$ in (9), it is found that $\beta = 0$, $\delta = 0$, $4\alpha\gamma + 1 = 0$. This gives a solution which, by a proper magnification, may be reduced to

$$y' = \frac{y^2 + x^2 - 1}{2x}; \tag{14}$$

the corresponding integral is

$$x + 2 \tan^{-1} \frac{1 - y}{x} = \text{const.} \tag{14'}$$

To make the discussion complete it is necessary to consider separately the assumption, not included in the form (10), that R is constant. This, however, gives only imaginary solutions

$$y' = a(x \pm iy)^2 + b(x \pm iy) + c. \tag{15}$$

The conclusions on page 346 should therefore read as follows:

If an equation of the Riccati type proper represents an isothermal system, it is reducible to one of the forms (14) *or* (15).

If a real isothermal system has a differential equation of the form $y' = P + Qy + Ry^2$, *it is similar to one of the five systems numbered* (3), (4), (6), (7), (14').

* My attention was called to the additional solutions by Mr. J. E. Wright who kindly communicated the results of an independent examination of the problem.

† In the original paper the last term in the value of P appears with a negative sign. There are two other errata on p. 345: the second term in the third line should be $-4RR'Q'$, and the comma at the end of the fourth line from below should be deleted.

COLUMBIA UNIVERSITY,
March, 1904.

SHORTER NOTICES.

Étude de quelques surfaces algébriques engendrées par des courbes du second et du troisieme ordre. Par M. STUYVAERT, Professeur à l'Athenée Royale de Gand. (Thèse). Gand, Hoste, 1902. 72 pp.

CONSIDERING surfaces generated by a curve, plane or twisted, as a logical generalization of the ruled surface, Stuyvaert discusses surfaces generated by a conic or a twisted cubic. His style is clear and concise and he arrives at many known results in a new way and gives many new and interesting ones. The dissertation is divided into three chapters. Chapter I discusses surfaces which are generated by a plane which contains a conic having six points on a directrix curve or system of directrix curves. The first case considered is that of a single directrix C_m, a unicursal curve of order m. The class of this surface is determined as follows. Let S_2 be a quadric passing through four fixed points, A, B, C, D, and cutting C_m in $2m$ points, and let π be a plane through a general line d, cutting C_m in m points. The class of the surface will be the number of groups of six points common to the planes through d and the quadrics through A, B, C, D. The quadrics mark on C_m an involution I_5^{2m}, the planes mark an involution I_1^m. The number of groups of six points common to the two is expressed by

$$C_1^{2m-5} \cdot C_5^{m-1} = (2m - 5)C_5^{m-1}.$$

But in each of the four planes (dA), (dB), (dC), (dD), there are C_5^m groups of six points situated on degraded quadrics and hence should be excluded, therefore the class is given by

$$\nu = (2m - 5)C_5^{m-1} - 4C_5^m.$$

By essentially the same method the class of surfaces whose directrix curves consist of a line and a C_m, a conic and C_m, etc., is obtained. As corollaries of the preceding theorems follow readily the determinations of the orders of surfaces generated by ∞^1 conics which move on a system of directrices. In this connection the theorem is given: *Conics whose plane passes through an axis $\alpha\beta$, and which cut five independent lines generate a surface of order eight.*

In chapter II surfaces generated by ∞^1 conics are discussed. First it is shown that the order of a surface generated by conics whose planes pass through an axis $\alpha\beta$ and which cut each of five unicursal directrices in one point depends upon the order of the surface which has five rectilinear directrices. This last surface is shown to be of order eight and is denoted by S_8.

The analytic discussion of the surface S_8 is then taken up. It is unicursal; for if through a point M of the surface a line be drawn cutting the axis $\alpha\beta$ and one directrix, it will pierce an arbitrary plane π in a point P, the image of M. Conversely a line drawn through P cutting the axis and one directrix cuts the surface in a single point M. The surface can therefore be depicted on the plane rationally. Two parameters can therefore be chosen, in terms of which the surface can be expressed rationally. The first is ω, so chosen that

$$\alpha_x + \omega\beta_x = 0$$

represents the plane $(\alpha\beta M)$, α_x, β_x being planes through the axis $\alpha\beta$. The second parameter is the anharmonic ratio λ of M, C, C', C'' where the generator through M cuts three directrices. In terms of these two parameters the coördinates of the surface are rationally expressed in terms of functions of degree seven in ω and degree two in λ. The remainder of the chapter is concerned principally with the properties of this surface and its plane depiction. Among others this interesting theorem is derived: *Every quadric passing through a generating conic of S_8 is tangent to the surface in four points of this conic.* This is an extension of a corresponding theorem for ruled surfaces.

Chapter III deals with sheaves of twisted cubics. A sheaf which contains the sheaf of Reye and the sheaf of Sturm as special cases are discussed in considerable detail. The sheaf is defined by

$$\alpha a_x/b_x = \alpha' a'_x/b'_x = \alpha'' a''_x/b''_x$$

where a_x, b_x, $a'_x \cdots$ are planes and α, α', α'' are arbitrary constants. It is seen that the cubics have in common three bisecants and two points. These equations will represent the sheaf of Reye, consisting of all the cubics passing through five points, if

$$|aba'b'| = |aba''b''| = |a'b'a''b''| = 0.$$

They will represent the sheaf of Sturm, consisting of all the cubics having the same osculating tetrahedron, if

$$a'_x \equiv b_x \quad a''_x \equiv b'_x.$$

Surfaces generated by ∞^1 of these cubics, *e. g.*, all those cutting a fixed line, touching a plane, etc., are discussed at some length. The plane depiction of these surfaces is also taken up. This chapter takes up nearly half the paper. Many interesting theorems concerning this sheaf are proved, also the corresponding ones for the sheaf of Reye and of Sturm are given. Several problems are suggested in the course of the discussion, some of which the author states he expects to solve.

C. L. E. MOORE.

Das Erdsphäroid und seine Abbildung. Von Dr. EMIL HAENTZSCHEL, Professor an der kgl. technischen Hochschule und am köllnischen Gymnasium zu Berlin. Leipzig, Teubner. 1903. viii + 140 pp., 16 figures.

THE purpose of this book is to discuss the practical problems of map drawing. It differs from many other works on the subject by leaving aside all those problems which are only of theoretical interest, and by including most of the numerical calculation of those considered. The author makes no claim for completeness, but still he presents enough of the subject to make his problem of the actual construction of geographical maps entirely intelligible. The book is very full of references to more extensive treatments of each particular problem discussed. A knowledge of the relations between exponential and trigonometric functions and of the elements of analytic geometry and the calculus is presupposed, although most of the formulas are derived with great detail. An introduction presents the evidence for the spheroidal form of the earth; it is assumed to be of revolution and Bessel's constants are used. The author mentions that probably Clarke's determination is more accurate than Bessel's.

The first chapter discusses the relations between the various kinds of latitude, geographic, geocentric and reduced (eccentric angle), and the determination of the maximum difference between them. The length of a degree along a meridian is fully discussed and it is clearly shown why a knowledge of its length is valuable. The area of a zone defined by two parallels of latitude is shown not to depict on the concentric sphere in

such a way as to preserve a constant ratio between corresponding areas. The theory of the large Prussian maps (Messtischblätter) with linear scale of 1 to 25,000 units is discussed, and elaborate calculations are given to show the variation between the lengths of north and of south boundaries between two common meridians.

The second chapter begins with a much more theoretical discussion of two spherical representations, that in which areas are preserved, and the conformal. The maximum difference between the spherical and the geographic latitude is determined. About forty pages are then devoted to the double conformal representation, first of the spheroid upon the gaussian sphere, then the latter upon the plane by Mercator's projection, the curve of contact of the sphere and cylinder being the principal meridian. The process is illustrated by actually determining four points upon the plane when their spheroidal coördinates are known by triangulation and astronomical observation. The calculation covers ten pages, every detail being given and all the numerical work being done with 7-place logarithms.

The book is almost free from typographical errors. On page 50 is a slip; an elementary integral is there called elliptic. The sections are marked by bold-faced type, but only named in the table of contents. The book is provided with an index.

VIRGIL SNYDER.

Die Horopterkurve, mit einer Einleitung in die Theorie der kubischen Raumkurve. Von Dr. WALTHER LUDWIG. Halle, Schilling, 1902. 36 pp.

THIS monograph was prepared to explain the new models of twisted cubic curves (Schilling's catalogue, series XXVII, number 6). Until recently the only models of these curves in the Schilling collection were those on plaster cylinders, which on account of the opaqueness of the material and the smallness of the scale conveyed but little information.

The present series consists of six pieces; four represent the standard forms of the curve traced on celluloid cylinders about twenty inches high, the fifth is the developable of class three made of silk threads and the sixth is a heavy wire model of the horopter.

The memoir is divided into two parts, the general discussion of the cubic curve, and the particular discussion of the horopter. The first part is very elementary. It begins with a dis-

cussion of projective pencils of planes. Then the cubic curve is shown to be the locus of intersection of three corresponding planes in three projective pencils. The generation of the same curve as locus of intersection of corrresponding lines of two projective bundles is treated more briefly and less clearly, although it is the only generation with which we are concerned in the second part. The usual classification of the curves according to the configuration on the infinite plane completes the first part.

The second part begins with the physiological principles underlying the horopter and their geometric meaning. The locus of the point which appears as a single point for a fixed position of the eyes is called the horopter of this position. In order to obtain the form of this curve various theorems are assumed which are only established empirically — those relating to the center of vision and center of rotation. The images on the retinas, connected with the center, form two projective bundles of rays. When two corresponding rays intersect, the point which they project is seen simply. The case considered is that in which the head is vertical, the line of sight horizontal and directly in front of the person.

The desired horopter is therefore a twisted cubic. It can be defined in terms of a parameter t as follows:

$$x = \frac{-2b \sin \psi t}{t^2 + 2t \cos \psi + 1}, \quad y = \frac{b(1 - t^2)}{t^2 + 2t \cos \psi + 1}, \quad z = \frac{c(1 - t)}{1 + t}.$$

It lies on the cylinder of revolution

$$x^2 + y^2 - 2bx \cot \psi - b^2 = 0.$$

Since two points on each secant of the curve are seen simply, it follows that every chord of the horopter appears as a simple line, though its various points may appear double. The curve passes through K_1, K_2, the vertices of the two projective bundles. The finite arc between K_1, K_2 has no physiological meaning.

In the model the curve itself is made of brass — a line of symmetry and an asymptote are of nickel. Two little spheres represent K_1, K_2; from these pass two copper wires to a third sphere on the curve. The copper wires represent lines of sight. Two white lines on the base of the stand supporting the curve indicate the intersections with this plane of the frontal and median planes.

VIRGIL SNYDER.

Introduction to the Theory of Algebraic Equations. By LEONARD EUGENE DICKSON, Ph.D., Assistant Professor of Mathematics in the University of Chicago. New York, John Wiley and Sons, 1903. Small 8vo, v + 104 pp.

THE aim of this little volume is to furnish a very elementary introduction to the Galois theory of algebraic equations. The book is divided into eleven brief chapters bearing the following headings: Solution of the general quadratic, cubic and quartic equations; Substitutions; Substitution groups and rational functions; The general equation from the group standpoint; Algebraic introduction to Galois theory: The group of an equation; Solution by means of resolvent equations; Regular cyclic equations, abelian equations; Criterion for algebraic solvability; Metacyclic equations, Galoisian equations; An account of more technical results.

There are five sets of exercises in addition to a number of illustrative examples. These examples are an especially commendable feature, as they are carefully selected and of a sufficiently elementary character to aid the student in avoiding the many pitfalls which beset his way when he first enters upon this difficult subject. In view of the fact that the literature of this subject contains so few usable examples which serve to illustrate some of the most difficult points, such as the group of a special equation, the present lists are especially welcome.

An appendix of four pages and a complete index close the volume. The former is devoted to Gauss's proof of the fundamental theorem on symmetric functions, and to Moore's proof of the theorem that a rational, integral function of $x_1, x_2, \cdots, x_n$ with constant indeterminate coefficients cannot be equal to zero unless it is identically zero. The proofs throughout are brief and the matters are stated in a clear and definite form. Possibly the brevity of proof has sometimes been carried to an extreme, as on top of page 18. However, the principal object of an introductory course should be to lead a student to a clear comprehension of the main questions and results, and the present volume seems admirably adapted to this end.

G. A. MILLER.

Elements of the Theory of Integers. By JOSEPH BOWDEN, Ph.D. New York, The Macmillan Company, 1903. x + 258 pp. Price $1.25.

PROFESSOR BOWDEN has undertaken in this volume to base the elementary theory of numbers on the three fundamental ideas of number, equality and sum using the following axioms only:

1. $a = a$.
2. If $a = b$ then $b = a$.
3. If $a = b$ and $b = c$ then $a = c$.
4. $a + b = b + a$.
5. If $a = b$ then $a \neq c + b$.
6. If $a \neq b$ there is some number c such that either $a = c + b$ or $c + a = b$.
7. If $a = b$ then $a + c = b + c$.
8. If $a \neq b$ then $a + c \neq b + c$.
9. $(a + b)\, c = a + (b + c)$.
10. Every number except unity is the sum of two numbers.

This list the author submits as a set of independent and sufficient axioms.

The book is divided into two parts to which the author hopes to add three more. The first part deals with the fundamental ideas, axioms etc., together with the statement of certain principles of logic to be used in developing the subject. The notion of subtraction is introduced together with the ideas "greater than," "less than" etc. In this part the difference $a - b$ is not considered as having a meaning unless $a > b$. Numbers in this first part are distinguished as *primary* numbers. In the second part an integer is defined as a "number couple" made by combining two primary numbers a and b, this number couple being the difference $a - b$, when $a > b$ and in all cases having properties like those of a difference. The fundamental operations of addition, subtraction, multiplication and division are defined for these number couples.

There is a chapter on factors in which the sieve of Eratosthenes is described, and the resolution of a composite number into its prime factors is discussed. Chapters are also devoted to the greatest common factor and to the least common multiple. The book closes with a chapter on congruences.

Professor Bowden makes use of a very elaborate and unusual system of notation. The greatest common factor of α and β

is written, $\alpha \otimes \beta$ and is read "α *kor* β." The least common multiple of α and β is written, $\alpha \times \beta$ and is read "α *mul* β." "α divisible by β" is written, $\alpha \succ \beta$ and "α prime to β" is written, $\alpha \amalg \beta$. This symbolism, together with a new and somewhat inconsistent system of orthography, gives the book an uninviting appearance and increases materially the difficulty of reading it.

D. N. LEHMER.

NOTES.

THE opening (January) number of volume 26 of the *American Journal of Mathematics* contains the following papers: "On primitive groups of odd order," by H. L. RIETZ; "Theorems on cardinal numbers," by A. N. WHITEHEAD; "The caustic, by reflection, of a circle," by T. J. I'A. BROMWICH; "On imprimitive substitution groups," by H. W. KUHN.

THE April number (volume 5, number 3) of the *Annals of Mathematics* contains: "On a system of hypocycloids of class three inscribed to a given three-line, and some curves connected with it," by H. A. CONVERSE; "Determination of all groups of binary linear substitutions with integral coefficients taken modulo 3 and of determinant unity," by L. E. DICKSON; "Projective and metric geometry," by E. B. WILSON; "A geometric discussion of the absolute convergence of a series with complex terms," by G. H. LING.

AT the meeting of the London mathematical society on March 10, the following papers were read: "On inner limiting sets of n points," by Dr. E. W. HOBSON; "On the unique expression of a quantic of any order in any number of variables, with an application to binary perpetuants," by Mr. P. W. WOOD; "The derivation of generalized Bessel functions from a function analogous to the exponential," by the Rev. F. H. JACKSON; "Illustrative examples of modes of decay of vibratory motions," by Professor A. E. H. LOVE.

The society has now 261 members and 13 honorary members. The officers for the present year are: president, Professor H. LAMB; vice-presidents, Professors E. B. ELLIOTT, E. W. HOBSON, H. F. BAKER; treasurer, Professor J. LARMOR; secretaries, Professors A. E. H. LOVE and W. BURNSIDE.

THE Deutsche Mathematiker-Vereinigung now numbers 608 members. The officers for the present year are: president, Professor H. WEBER; secretary, Professor A. KRAZER. Two new members have been elected to the council: Professors H. VON MANGOLDT and P. STÄCKEL. The next regular meeting will be held in Breslau, September 18 to 24, 1904, in connection with the seventy-sixth general meeting of the society of German naturalists and physicians.

ON January 14, 1904, a new mathematical society was founded at Vienna, with Professor G. VON ESCHERICH as president and Professor A. LAMPE as secretary. No publication of its proceedings will be undertaken for the present.

AT the second international congress of philosophy, to meet at Geneva September 4–8, 1904, communications from the following are announced in the section on the history of sciences (the third international congress of the history of sciences): Baron CARRA DE VAUX, Professors DUHEM, HANNEQUIN, MILHAUD, P. TANNERY, and ZEUTHEN, and Dr. R. BLANCHARD.

AT the second regular meeting of the association of teachers of mathematics in the Middle States and Maryland, held at Columbia University April 2, 1904, the following papers were presented: — By Dr. J. T. RORER: "Has algebra any genuine applications (*a*) to the student's present interests and experiences, (*b*) to his probable future needs? If so, what are they and how can they be introduced?" — By Professor G. LEGRAS: "The great movements now taking place in the teaching of mathematics in other countries." — By Professor D. E. SMITH: "Historic methods of calculation" (Illustrated by stereopticon reproductions of manuscripts and early books.)

AT a conference held in Columbus, Ohio, February 22, 1904, steps were taken to perfect an organization of Ohio teachers of mathematics and the sciences. The organization was completed at a meeting held April 2, at which the following officers were elected: president, Professor C. G. HOWE; secretary, Professor T. E. MCKINNEY. The objects of the association are to improve the teaching of mathematics and the sciences in Ohio and to strive for uniform entrance requirements at the various colleges. The next meeting will be held at Columbus in April, 1905.

THE editors of the *Monatshefte der Mathematik und Physik* for the present year are Professors G. VON ESCHERICH, F. MERTENS and W. WIRTINGER.

L'Enseignement Mathématique is now edited by Professors C. A. LAISANT, H. FEHR and A. BUHL. The entire business management has been transferred to Geneva, Switzerland, and will be in charge of Professor Fehr.

UNDER the title of *Mathematisch-Naturwissenschaftliche Blätter*, a monthly journal has been founded by the general association of students of mathematics in the German universities. The first number appeared in January.

THE following works are announced as in the press of B. G. Teubner in Leipzig: Jules Drach, Histoire des sciences mathématiques en France au XIX^{e} siècle, 8° (about 320 pages); H. Poincaré, Wissenschaft und Hypothese, translated by Professor F. Lindemann; E. Netto, Elementare Algebra; C. F. Gauss, Werke, volume 7, Theoria motus und astronomischer Nachlass, edited by Professor Brendel.

A RECENT prospectus from the publishing house of Gauthier-Villars in Paris announces that the following mathematical works are in the press and will shortly appear: Andoyer, Theorie des formes, volume 2; Goursat, Cours d'analyse mathématique, volume 2, containing the theory of analytic functions, ordinary and partial differential equations and the elements of the calculus of variations; Humbert, Cours d'analyse, volume 2, containing the completion of the theory of definite integrals, Euler's functions, functions of a complex variable, elliptic functions with their applications, and differential equations.

A TREATISE on the calculus of variations by Professor HARRIS HANCOCK will shortly be published by the University of Cincinnati.

THE UNIVERSITY OF CHICAGO. — The following advanced courses in pure and applied mathematics will be given during the summer quarter. — By Professor O. BOLZA: Advanced integral calculus; Multiple algebra and quaternions. — By Professor H. MASCHKE: Partial differential equations; Projective geometry. — By Professor H. E. SLAUGHT: Advanced algebra. — By Professor J. W. A. YOUNG: Conferences on the pedagogy of mathematics. — By Professor K. LAVES: Analytic mechanics. — By Professor F. R. MOULTON: The problem of three

bodies. — By Professor G. W. MYERS: Pedagogy of elementary mathematics; Pedagogy of secondary mathematics. Each course will be given five hours weekly.

The American universities below offer during the academic year 1904–1905 advanced courses in mathematics as follows:

HARVARD UNIVERSITY. — By Professor J. M. PEIRCE: Rigid dynamics; Triangular coördinates and higher plane curves; Quaternions (second course). — By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and the potential function. — By Professor B. O. PEIRCE: The mathematical theory of electricity and magnetism. — By Professor W. F. OSGOOD: Calculus (second course); † Infinite series and products; and *either* † Galois's theory of equations, *or* † Riemann's theory of functions. — By Professor M. BÔCHER: Introduction to modern geometry and modern algebra; The theory of functions; Linear differential equations, total and partial. — By Professor C. L. BOUTON: Differential equations and Lie's theory. — By Mr. J. L. COOLIDGE: The geometry of position. — By Mr. J. K. WHITTEMORE: Differential geometry; † The theory of the form and the rotation of the planets.

These courses will involve three lectures a week throughout the academic year except those preceded by †, which involve three lectures a week for half the year. Courses in reading and research, which differ from the foregoing in involving no lectures on the part of the instructor, will also be offered as follows: By Professor J. M. PEIRCE: Linear associative algebrá. — By Professor BYERLY: Recent contributions to the ellipsoidal harmonic analysis.— By Professor BÔCHER: Differential equations. — By Professor BOUTON: Continuous groups. — By Mr. COOLIDGE: Higher geometry. — By Mr. WHITTEMORE: Goursat's Cours d'analyse.

YALE UNIVERSITY. — By Professor J. PIERPONT: Projective geometry, three hours; Functions of a complex variable, three hours; Theory of numbers, three hours. — By Professor P. F. SMITH: Advanced differential geometry, three hours. — By Professor H. E. HAWKES: Geometry, two hours; Determinants and invariants, two hours. — By Dr. W. A. GRANVILLE: Analytic geometry, three hours. — By Dr. L. P. WHEELER: Theory of electrons, one hour. — By Dr. E. B.

WILSON: Advanced calculus, two hours; Introduction to mathematical physics, two hours; Electromagnetic theory of light, two hours. — By Dr. A. S. GALE: Analytic mechanics, two hours; Differential equations, two hours. — By Dr. C. M. MASON: Calculus of variations, three hours. — By Dr. D. R. CURTISS: Ordinary differential equations, three hours. — By Mr. TAYLOR: Scientific computation, two hours.

THE foreign universities below offer courses in mathematics during the summer of 1904 as follows:

OXFORD UNIVERSITY. — The following courses in mathematics are announced for the Easter and Trinity terms, 1904: By Professor W. ESSON: Comparison of analytic and synthetic methods in the geometry of conics II, two hours; Informal instruction in geometry, one hour. — By Professor E. B. ELLIOT: Theory of functions, three hours. — By Professor A. E. H. LOVE: Geometric optics, two hours; Higher geometric optics, one hour. — By Mr. A. L. DIXON: Calculus of variations, one hour. — By Mr. J. E. CAMPBELL: An introduction to the theory of groups, one hour. — By Mr. P. J. KIRKBY: Higher plane curves, one hour. — By Mr. H. T. GERRANS: Line geometry, two hours. — By Mr. A. E. JOLLIFFE: Higher analytic plane geometry, two hours. — By Mr. J. W. RUSSELL: Rigid dynamics, three hours — By Mr. A. L. PEDDER: Spherical trigonometry, one hour. — By Mr. R. F. MCNEILE: Series, two hours. — By Mr. C. H. THOMPSON: Differential equations, two hours. — By Mr. C. E. HASELFOOT: Elementary hydrostatics, one hour.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Synthetic geometry, four hours; Selected problems in conformal representation, two hours; Theory of analytic functions II, two hours; Colloquium, one hour. — By Professor G. FROBENIUS: Analytic geometry, two hours. — By Professor F. SCHOTTKY: Algebraic analysis, four hours; Theory of curves and surfaces, four hours. — By Professor R. LEHMANN-FILHÉS: Analytic mechanics, four hours; Problem of three bodies, two hours. — By Professor J. KNOBLAUCH: Integral calculus, four hours; with exercises, one hour; Applications of elliptic functions, four hours. — By Dr. E. LANDAU: Differential calculus, four hours; with exercises, one hour; Theory of irrational numbers, one hour; Exercises in theory of func-

tions, one hour. — By Dr. I. SCHUR: Theory of algebraic equations, four hours; Theory of substitutions, two hours; with exercises, two hours.

UNIVERSITY OF BONN. — By Professor H. KORTUM: Theory of determinants, two hours; Theory of numbers, four hours; Seminar, two hours. — By Professor L. HEFFTER: Differential and integral calculus, four hours; with exercises, one hour; Theory of elliptic functions, four hours. — By Professor E. STUDY: Analytic geometry I, four hours, with exercises, two hours; Curves of the fourth order, two hours; Seminar, two hours. — By Professor J. SOMMER: Introduction to the theory of differential equations, three hours.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Plane analytic geometry, four hours; Elements of determinants, two hours; Seminar, one hour. — By Professor R. STURM: Differential geometry, three hours; Higher geometric loci, three hours; Seminar two hours. — By Professor F. LONDON: Theory of elliptic functions, four hours.

UNIVERSITY OF ERLANGEN. — By Professor P. GORDAN: Differential equations, four hours; Theory of numbers, four hours; Seminar, three hours. — By Professor M. NOETHER: Analytic geometry of space, four hours; Synthetic geometry with exercises, four hours; Exercises in geometry, one hour.

UNIVERSITY OF FREIBURG — By Professor J. LÜROTH: Integral calculus, five hours; with exercises, two hours; Plane and spherical trigonometry, two hours. — By Professor L. STICKELBERGER: Mechanics, five hours; Calculus of variations, three hours; Seminar, two hours. — By Professor A. LOEWY: Theory and application of determinants, three hours; History of mathematics, two hours; Introduction to higher mathematics with applications to natural sciences, two hours. — By Dr. SEITH: Descriptive geometry, two hours; with exercises, one hour.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Algebra, four hours; Determinants, two hours; Seminar, one hour. — By Professor E. NETTO: Plane analytic geometry, four hours; Theory of numbers, two hours; Seminar, one hour.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ, Alegebra, four hours; Synthetic geometry, two hours; Semi-

nar, two hours. — By Professor G. KOWALEWSKI: Mechanics I, four hours; with exercises, one hour: Applications of elliptic functions, one hour. — By Professor L. EBERT: Theory of probabilities and the method of least squares, one hour; Fundamental equations of the theory of perturbations, one hour.

UNIVERSITY OF HEIDELBERG. — By Professor L. KOENIGSBERGER: Differential and integral calculus, four hours; Theory of curves and surfaces, four hours; Seminar, two hours. — By Professor M. CANTOR: Analytic geometry of the plane, four hours; Arithmetic and algebra, three hours. — By Professor F. EISENLOHR: Theory of probabilities, three hours; Mechanics, four hours. — By Professor K. KOEHLER: Plane synthetic geometry, three hours. — By Professor G. LANDSBERG: Theory of functions, four hours; Theory of determinants, two hours. — By Dr. K. BOEHM: Elementary mathematics, three hours.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Introduction to the theory of determinants, four hours; Algebraic curves and surfaces, four hours; Seminar, one hour. — By Professor P. HARZER: Exercises in numerical calculation, one hour. — By Professor P. STÄCKEL: Differential calculus and introduction to higher analysis, four hours; Higher algebra, four hours; Calculus of variations, two hours; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG. — By Professor F. MEYER; Theory of numbers, four hours; Seminar, in connection with Professors Volkmann and Schoenflies, two hours. — By Professor A. SCHOENFLIES: Analytic geometry, five hours; Seminar, two hours. — By Professor L. SAALSCHÜTZ: Differential calculus, four hours; Pseudo-elliptic integrals of the third kind, one hour. — By Dr. F. COHN: Dynamics, according to the methods of Hamilton and Jacobi, three hours. — By DR. T. VAHLEN: Algebra, with exercises, five hours.

UNIVERSITY OF LEIPZIG. — By Professor A. MAYER: Partial differential equations of the first order, four hours. — By Professor O. HÖLDER: Theory of algebraic equations, four hours; On the foundations of arithmetic, two hours. — By Professor F. ENGEL; Ordinary differential equations, four hours; Theory of functions, two hours; Theory of transformations and

application to differential equations, two hours; Seminar, two hours. — By Professor F. HAUSDORFF; Introduction to analytic geometry with exercises, four hours; Noneuclidean geometry, two hours. — By Dr. H. LIEBMANN: Differential geometry, with exercises, four hours.

UNIVERSITY OF MARBURG. — By Professor K. HENSEL: Theory of functions, four hours; Theory of partial differential equations and their application to problems in physics, three hours; Selected chapters of algebra (theory of groups), one hour; Seminar, two hours. — By Dr. F. JUNG: Determinants, two hours; Theory of probabilities, two hours.

UNIVERSITY OF MÜNSTER. — By Professor L. KILLING: Analytic geometry I, four hours; Theory of functions, four hours; Seminar, two hours. — By Professor R. VON LILIENTHAL; Analytic mechanics, four hours; Selected chapters of differential geometry, two hours; Theory of determinants, two hours; Seminar, two hours. — By Dr. M. DEHN: Differential and integral calculus, four hours; Algebra II, two hours; exercises, two hours.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: Introduction to synthetic geometry, two hours; Technical mechanics, four hours; Seminar, two hours. — By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Partial differential equations of mathematical physics, four hours; Seminar, one and one half hours. — By Professor M. SIMON: Methods of elementary arithmetic in connection with algebraic analysis, two hours. — By Professor F. ROTH: Differential and integral calculus, three hours; with exercises, one hour; Plane analytic geometry, three hours. — By Professor M. DISTELI: Analytic geometry of space, three hours; Perspective with exercises, three hours; Seminar, one and one half hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. VON BRILL: Mechanics, five hours; Twisted curves and surfaces, two hours; Seminar, two hours. — By Professor H. STAHL: Elementary analysis, three hours; Theory of functions, three hours; Seminar, two hours. — By Professor L. MAURER: Higher analysis, three hours; Synthetic geometry, two hours; with exercises, two hours.

UNIVERSITY OF BASEL (From April 19 to August 16.) —

By Professor H. KINKELIN: Differential and integral calculus II, three hours; Algebraic analysis, three hours; Projective geometry, three hours.

UNIVERSITY OF BERN (From April 19 to July 23).— By Professor J. H. GRAF: Spherical harmonics, three hours; Bessel functions, three hours; Gamma functions and Bernoulli functions, three hours; Differential equations II, two hours; Differential and integral calculus, two hours; Calculation of investments and insurance, two hours; Seminar (with Professor Huber), two hours; Seminar (insurance, with Professor Moser), two hours. — By Professor G. HUBER: Orbits of multiple stars, one hour; Theory of plane curves, two hours; Cubature of solids and complanation of surfaces, one hour. — By Professor E. OTT: Differential calculus, two hours; Analytic geometry, two hours; Differential equations, two hours. — By DR. A. BENTELI: Descriptive geometry, four hours; with exercises, three hours. — By DR. L. CRELIER: Synthetic geometry, two hours; Kinematic geometry, two hours.

UNIVERSITY OF GENEVA (From April 8 to July 15).— By Professor C. CAILLER: Differential and integral calculus, three hours; Conference, two hours. — By Professor H. FEHR; Descriptive and projective geometry, two hours; Theory of equations, two hours; Seminar, in higher geometry, one hour; Exercises in the calculus (with Professor Cailler) two hours, in rational mechanics, two hours and in algebra and geometry, two hours. — By Dr. D. MIRIMANOFF: Equations of mathematical physics, two hours.

UNIVERSITY OF ZÜRICH. — By Professor H. BURKHARDT: Algebraic analysis, three hours; Differential and integral calculus II, three hours; Differential equations of physics, three hours; Seminar, two hours; By Professor A. WEILER: Descriptive geometry II, with exercises, four hours; Analytic geometry II, with exercises, three hours; Synthetic geometry, two hours; Political arithmetic II, two hours. — By Dr. F. GUBLER: Methods of instruction in secondary schools, two hours; Algebraic analysis, two hours; Exercises in political arithmetic, two hours.

THE mathematical and natural sciences section of the Prince Jablonowski society in Leipzig proposes the following prize problem for the year 1906:

"By recent investigations of Hurwitz, Matter, Krause and others the theory of the Bernoulli numbers and functions has been remarkably extended. As a continuation of these researches is believed to be important not only for the general theory of functions but for the theory of numbers and in particular for elliptic functions, the society proposes the following problem, without restricting the aim of possible competitors: An investigation of numbers analogous to the Bernoulli numbers, particularly in the field of elliptic functions, which admit complex multiplication."

Competing manuscripts should be sent to the secretary under the usual conditions before November 30, 1906. The prize is 1000 marks.

THE Royal academy of sciences of Madrid proposes the following prize problem for 1905:

"To give a complete theory of a class of special singular integrals of differential equations for which the derivatives become indeterminate when certain relations exist between the principal variables."

Competing memoirs should be written in Spanish or Latin and should be in the hands of the secretary of the academy before December 31, 1905.

CAMBRIDGE UNIVERSITY.—The Smith's prizes have been adjudged to E. CUNNINGHAM, for his essay "On the normal series satisfying linear differential equations," to C. M. GARNETT, for his essay "On the cause of metal glasses and metallic films," to H. A. WEBB, for his essay "On the expansion of an arbitrary function in a series of functions" and to P. W. WOOD, for his essay "On covariant types."

PROFESSORS J. A. GMEINER, of the German University of Prague, and K. ZINDLER, of the University of Innsbruck, have been promoted to full professorships.

PROFESSOR H. STAHL, of the University of Tübingen, has been awarded the order of the crown of Würtemburg.

PROFESSOR E. LAMPE, of the Technical high school of Berlin, has received the order of the crown of the third class from the German emperor.

PROFESSOR G. MITTAG-LEFFLER has recently been elected a foreign member of the Società Italiana delle Scienze, and honorary member of the Royal Irish Academy.

PROFESSOR P. STÄCKEL, of Kiel, has declined a call to the University of Marburg.

PROFESSOR G. H. DARWIN, of Cambridge, Professor C. SEGRE, of Turin, and Professor M. LÉVY, of Paris, have been elected foreign associates of the Belgian academy of sciences.

PROFESSOR F. ENGEL, of Leipzig, has been appointed professor of mathematics in the University of Greifswald.

PROFESSOR R. F. ALLARDICE, of Stanford University, has been granted leave of absence for the year 1904–1905, and expects to spend the year at European universities.

AT Harvard University, Dr. C. L. BOUTON has been promoted to an assistant professorship of mathematics.

DR. S. E. SLOCUM has been promoted to an assistant professorship in applied mathematics at the University of Cincinnati.

AT Stanford University, Dr. H. C. MORENO has been promoted to an assistant professorship in applied mathematics. Mr. W. A. MANNING, instructor in applied mathematics, has been granted a leave of absence. He will spend next year in European universities. His position has been filled by the appointment of Mr. G. I. GAVETT.

MESSRS. S. C. HAIGHT and C. E. MORRISON have been appointed tutors in mathematics in the College of the City of New York.

DR. C. M. MASON, of the Massachusetts Institute of Technology, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

DR. D. R. CURTISS, who has been spending the year at Paris, has been appointed instructor in mathematics at Yale University.

RECENT catalogues of second-hand books on mathematics: E. Loescher & Cie, 307 Corso Umberto I, Rome, catalogues nos. 66, 67, 69, 2003 titles of works on mathematics, physics, and astronomy, printed before 1800, and journals; H. Welter, rue Bernard-Palissy, 4, Paris, catalogue of February 10, 1904, chiefly journals; H. Kerler, Ulm, catalogue no. 322, 1356 titles; A. Hermann, rue de la Sorbonne, 6, Paris, V, catalogue no. 80, 2336 titles on mathematics, physics, and astronomy;

Krüger and Co., Leipzig, catalogue, no. 39, 1000 titles on mathematics and natural history; F. Strobel, Markt, 4, Jena, catalogue no. 17, 121 titles of scientific journals.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Niccolò Fergola. Napoli, 1903. 8vo. 32 pp.

BOHLMANN (G.). See SERRET (J. A.).

FORT (O.) und SCHLÖMILCH (O.). Lehrbuch der analytischen Geometrie. Teil I: Analytische Geometrie der Ebene, von O. Fort. 7te Auflage, besorgt von R. Heger. Leipzig, Teubner, 1904. 8vo. 17 + 268 pp. M. 4.00

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